Simplicity of Cuntz-Pimsner algebras

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1 Introduction

Given a $\mathcal{C}^*$-correspondence $(E, \epsilon)$ over a $\mathcal{C}^*$-algebra $A$, one may construct a certain $\mathcal{C}^*$-algebra $\mathcal{O}_E$ known as the Cuntz-Pimsner algebra (cf. [7]). This construction generalizes for instance the Cuntz algebras, Cuntz-Krieger algebras and crossed products over $\mathbb{Z}$ by a single automorphism. There are various results pertaining to simplicity of Cuntz-Pimsner algebras. This paper contains a detailed exposition of one of the more general results regarding simplicity, namely

[10, Theorem 3.9] For a full $\mathcal{C}^*$-correspondence $(E, \epsilon)$ over a unital $\mathcal{C}^*$-algebra $A$ with $\epsilon$ faithful, the Cuntz-Pimsner algebra $\mathcal{O}_E$ is simple if and only if $E$ is minimal and nonperiodic.

In addition to giving a thorough development of the material which culminates in the above main result, we consider some recently introduced examples of Cuntz-Pimsner algebras arising from self-similar group actions. The paper is organized as follows. In section 2 we give the basic terminology and results regarding $\mathcal{C}^*$-correspondences, the construction of the Cuntz-Pimsner algebra and crossed products over Hilbert $\mathcal{C}^*$-bimodules. Section 3 discusses Morita equivalence between crossed products over bimodules, granted Morita equivalence between the underlying $\mathcal{C}^*$-algebras. Here we also address Takai duality, i.e. determining the dual of the crossed product. The ideal structure of crossed products over bimodules is analyzed in section 4, where, due to the Takai duality result achieved in section 3, one is able to introduce the Connes spectrum into the study of the ideal structure. We are then able to prove the simplicity result for crossed products over equivalence bimodules, i.e. characterizing simplicity in terms of certain properties of the bimodule. The main result on simplicity of the Cuntz-Pimsner algebra $\mathcal{O}_E$ is then achieved in section 5 by passage to the simplicity result for the crossed product previously handled. In section 6 we apply the simplicity result to the Cuntz-Pimsner algebra $\mathcal{O}_\Phi$, where $\Phi$ is a $\mathcal{C}^*$-correspondence encoding the self-similar action of a group $G$ on a sequence space, as introduced in [2]. Whereas in [2], simplicity of this Cuntz-Pimsner algebra was handled in a way similar to that of the Cuntz algebra case, we deduce simplicity by showing minimality and nonperiodicity of the module, thus producing an alternative proof.

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2 Basic definitions and results

A $C^*$-correspondence $(E, \epsilon)$ over a $C^*$-algebra $A$ is a (right) Hilbert $A$-module $E$ together with a *-homomorphism $\epsilon : A \to \mathcal{L}(E)$. The module $E$ is thus equipped with a left action by $A$, namely we define $a \cdot \xi = \epsilon(a)\xi$, for $a \in A, \xi \in E$. We shall assume the left action to be nondegenerate.

For any $C^*$-algebra $A$, the identity correspondence over $A$ consists of $A$ considered as a Hilbert $A$-module, i.e. $\langle x, y \rangle_A = x^*y$ for $x, y \in A$, $x \cdot a = xa$ and $a \cdot x = ax$ being the usual algebra multiplications.

A $C^*$-correspondence $(E, \epsilon)$ is said to be full if $\langle E, E \rangle_A = A$.

Definition 2.1. For $C^*$-correspondences $(E, \epsilon)$ over $A$ and $(F, \phi)$ over $B$, a semicovariant homomorphism $\pi : (E, \epsilon) \to (F, \phi)$ is a pair of maps $\pi = (\pi_A, \pi_E)$, where $\pi_A : A \to B$ is a *-homomorphism, and $\pi_E : E \to F$ is linear, such that for all $a, b \in A$ and $\xi, \eta \in E$,

$$
\pi_E(a\xi) = \pi_A(a)\pi_E(\xi), \quad \pi_E(\xi b) = \pi_E(\xi)\pi_A(b), \quad (1)
$$

$$
\langle \pi_E(\xi), \pi_E(\eta) \rangle_B = \pi_A(\langle \xi, \eta \rangle_A). \quad (2)
$$

We call a semicovariant homomorphism $\pi = (\pi_A, \pi_E)$ nondegenerate when $\pi_A$ is nondegenerate.

For a given $C^*$-correspondence $(E, \epsilon)$, we shall construct a certain $C^*$-algebra which will be universal with respect to semicovariant homomorphisms from $(E, \epsilon)$ to identity correspondences, i.e. any such semicovariant homomorphism will factor through this $C^*$-algebra.

Proposition 2.2. Let $(E, \epsilon)$ be a $C^*$-correspondence over $A$. There exists a $C^*$-algebra $A \rtimes_E \mathbb{N}$ and a nondegenerate semicovariant homomorphism $i : (E, \epsilon) \to A \rtimes_E \mathbb{N}$ such that for every semicovariant homomorphism $\pi : (E, \epsilon) \to B$ to any identity correspondence $B$, there exists a unique *-homomorphism $\pi_A \times \pi_E : A \rtimes_E \mathbb{N} \to B$ with $\pi_A = (\pi_A \times \pi_E) \circ i_A$ and $\pi_E = (\pi_A \times \pi_E) \circ i_E$.

Proof. Denote by $C$ the universal *-algebra generated by

$$
\{T_\xi, T_a : \xi \in E, a \in A\}
$$
subject to the following relations:

\[ a \mapsto T_a \quad \text{is to be a } \ast\text{-homomorphism}, \]

\[ \xi \mapsto T_{\xi} \quad \text{is to be a linear map}, \]

\[ T_{a\xi b} = T_a T_{\xi} T_b \quad \text{for } a, b \in A, \ \xi \in E, \]

\[ T_{\xi}^* T_{\eta} = T_{\langle \xi, \eta \rangle_A} \quad \text{for } \xi, \eta \in E. \]

Define

\[ ||T|| = \sup_{\rho} \{ \rho(T) : \rho \text{ C*-seminorm on } C \}, \quad T \in C. \]

The supremum is bounded, since for any C*-seminorm \( \rho \) on \( C \), the assignment \( a \mapsto \rho(T_a) \) defines a C*-seminorm on \( A \), whence \( \rho(T_a) \leq ||a|| \). Also \( \rho(T_{\xi})^2 = \rho(T_{\xi}^* T_{\xi}) = \rho(T_{\langle \xi, \xi \rangle_A}) \leq ||\langle \xi, \xi \rangle_A|| = ||\xi||^2 \). Denote by \( A \rtimes_{\varepsilon} E \) the completion of \( C \) with respect to this norm (after passing to the appropriate quotient). Then define the semicovariant homomorphism

\[ i : (E, \varepsilon) \longrightarrow A \rtimes_{\varepsilon} E \]

\[ i_A(a) = T_a \]

\[ i_E(\xi) = T_{\xi}. \]

For any given C*-algebra \( B \), which is considered as an identity correspondence over itself, and semicovariant homomorphism

\[ \pi = (\pi_A, \pi_E) : (E, \varepsilon) \longrightarrow B, \]

we define the \( \ast \)-homomorphism \( \pi_A \times \pi_E : A \rtimes_{\varepsilon} E \longrightarrow B \) on the generators of \( C \),

\[ \pi_A \times \pi_E(T_a) = \pi_A(a), \]

\[ \pi_A \times \pi_E(T_{\xi}) = \pi_E(\xi) \]

and extend it in the obvious way to the completion. It is evident from our construction that \( \pi_A \times \pi_E \circ i_A = \pi_A \) and \( \pi_A \times \pi_E \circ i_E = \pi_E. \)

**Remark 2.3.** We identify \( A \) with \( i_A(A) \subseteq A \rtimes_{\varepsilon} E \) and \( E \) with \( i_E(E) \subseteq A \rtimes_{\varepsilon} E \). One also uses the faithful Fock representation to understand the C*-algebra \( A \rtimes_{\varepsilon} E \) as operators on the one-sided Fock space \( F^1(E) = \oplus_{n \geq 0} E^\otimes n \), where \( E^\otimes 0 = E \otimes_{\varepsilon} \cdots \otimes_{\varepsilon} E, \ E^\otimes 0 = A \). The Fock representation is then defined on elementary tensors as

\[ \lambda_A(a) \xi_1 \otimes \cdots \otimes \xi_k = a \xi_1 \otimes \cdots \otimes \xi_k, \]

\[ \lambda_E(\xi) \xi_1 \otimes \cdots \otimes \xi_k = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_k, \]

for \( a \in A, \ \xi \in X \), and their adjoint operators are \( \lambda_A(a)^* = \lambda_A(a^*) \) and

\[ \lambda_E(\xi)^* \xi_1 \otimes \cdots \otimes \xi_k = \langle \xi, \xi_1 \rangle_A \xi_2 \otimes \cdots \otimes \xi_k. \]
We see that \((\lambda_A, \lambda_E) : (E, \epsilon) \rightarrow \mathcal{L}(F^1(E))\) is a semicovariant homomorphism,

\[
\begin{align*}
\lambda_E(a\zeta b)\xi_1 \otimes \cdots \otimes \xi_k &= a\zeta b \otimes \xi_1 \otimes \cdots \otimes \xi_k \\
&= a\zeta \otimes b\xi_1 \otimes \cdots \otimes \xi_k \\
&= \lambda_A(a)\lambda_E(\xi)\lambda_A(b)\xi_1 \otimes \cdots \otimes \xi_k.
\end{align*}
\]

\[
\langle \lambda_E(\eta), \lambda_E(\zeta) \rangle_{\mathcal{L}(F(X))} = \lambda_E(\eta)^*\lambda_E(\zeta),
\]

\[
\lambda_E(\eta)^*\lambda_E(\zeta)\xi_1 \otimes \cdots \otimes \xi_k = \lambda_E(\eta)^*\zeta \otimes \xi_1 \otimes \cdots \otimes \xi_k
\]

\[
= \langle \eta, \zeta \rangle_A \xi_1 \otimes \cdots \otimes \xi_k
\]

\[
= \lambda_A(\langle \eta, \zeta \rangle_A)\xi_1 \otimes \cdots \otimes \xi_k.
\]

thus by Proposition 2.2 \((\lambda_A, \lambda_E)\) factors through \(A \rtimes E \mathbb{N}\), inducing the unique 
\(*\)-homomorphism \(\lambda_A \times \lambda_E : A \rtimes E \mathbb{N} \rightarrow \mathcal{L}(F^1(E))\) with \(\lambda_A = \lambda_A \times \lambda_E \circ i_A\)
and \(\lambda_E = \lambda_A \times \lambda_E \circ i_E\).

Later we shall put to use the Fock representation on the two-sided Fock space, and also establish faithfulness of the representation (Remark 2.13).

Given a semicovariant homomorphism \(\pi : (E, \epsilon) \rightarrow (F, \phi)\) we can define a \(*\)-homomorphism \(\pi_K(E) : K(E) \rightarrow K(F)\) by

\[
\pi_K(E)(\theta_{\xi, \eta}) = \theta_{\pi_E(\xi), \pi_E(\eta)}.
\]

**Definition 2.4.** For C*-correspondences \((E, \epsilon)\) and \((F, \phi)\) over a C*-algebra \(A\), a semicovariant homomorphism \(\pi = (\pi_A, \pi_E) : (E, \epsilon) \rightarrow (F, \phi)\) is called a **covariant homomorphism** if

\[
\phi \circ \pi_A = \pi_K(E) \circ \epsilon \quad \text{holds on } \epsilon^{-1}(K(E)).
\]

We then commence by constructing a universal C*-algebra for covariant homomorphisms to identity correspondences.

**Proposition 2.5.** Let \((E, \epsilon)\) be a C*-correspondence over \(A\). There exists a C*-algebra \(O_E\) and a nondegenerate covariant homomorphism \(j : (E, \epsilon) \rightarrow O_E\) such that for every covariant homomorphism \(\pi : (E, \epsilon) \rightarrow B\) to any identity correspondence \(B\), there exists a unique \(*\)-homomorphism \(\hat{\pi} : O_E \rightarrow B\) with \(\pi_A = \pi \circ j_A\) and \(\pi_E = \hat{\pi} \circ j_E\).
Proof. Continuing with the construction from the proof of Proposition 2.2, we use the map $i_{K(E)} : K(E) \to A \rtimes E \mathbb{N}$

$$i_{K(E)}(\theta_{\xi,\eta}) = T_{\xi} T_{\eta}^*,$$

and define $J$ to be the ideal in $A \rtimes E \mathbb{N}$ generated by

$$\{ i_{K(E)} \circ \epsilon(a) - i_A(a) : a \in \epsilon^{-1}(K(E)) \}.$$ 

The Cuntz-Pimsner algebra is $O_E = A \rtimes E \mathbb{N} / J$. Denoting the quotient map $q : A \rtimes E \mathbb{N} \to O_E$ and putting $S_\xi = q(T_{\xi})$, $S_a = q(T_a)$, gives the canonical covariant homomorphism $j = (j_A, j_E) : (E, \epsilon) \to O_E$, $j_A = q \circ i_A$, $j_E = q \circ i_E$, and furthermore $j_{K(E)} = q \circ i_{K(E)}$. The generators in $O_E$ thus satisfy the relations

$$S_{\alpha \xi b} = S_a S_\xi S_b, \quad S_\xi S_\eta = S_{(\xi,\eta)\Lambda}, \quad \text{for } a, b \in A, \xi, \eta \in E,$n

and for any $\zeta \in H$ we have

$$S_{\xi_1} S_{\xi_2} = S_{\xi_1, \xi_2} = S_{\delta_{1,2}}.$$

Any $\zeta \in H$ has an expansion with respect to the orthonormal basis, i.e. $\zeta = \sum_i \langle \xi_i, \zeta \rangle$, thus we get

$$S_{\xi_1} S_{\xi_2} = S_{\xi_1, \xi_2} = S_{\sum_i \langle \xi_i, \zeta \rangle} = S_{\zeta}.$$

It follows that $S_{\xi_1} S_{\xi_2}^*$ is the unit, denote it $I$, and by putting $S_\zeta = S_{\xi_1}$ we can express the above in the more aesthetic and familiar form

$$S_\zeta S_\zeta^* = I, \quad \zeta = 1, \ldots, n.$$

which are the Cuntz algebra relations, hence $O_H \cong \mathcal{O}_n$. 

Example 2.6. Let $H$ be a Hilbert space of dimension $n \geq 2$. If we choose some orthonormal basis $\{\xi_i\}_i$, then $O_H$ is generated by elements $\{S_{\xi_i}\}_i$ and $j_{K_0}(\mathbb{C})$, and we see from the relations that

$$S_{\xi_1} S_{\xi_2} = S_{\xi_1, \xi_2} = S_{\delta_{1,2}}.$$
Example 2.7. The Cuntz-Krieger algebra \( \mathcal{O}_B \) for any given \( n \times n \) matrix \( B = (b_{ij})_{i,j} \) with entries \( b_{ij} \in \{0,1\} \) and each column and row nonzero, is defined as the C*-algebra generated by partial isometries \( T_1, \ldots, T_n \) whose support projections \( Q_i = T_i^*T_i \) and range projections \( P_i = T_iT_i^* \) satisfy the relations

\[
P_iP_j = 0, \text{ for } i \neq j, \quad \text{and} \quad Q_i = \sum_{j=1}^n b_{ij}P_j, \quad i = 1, \ldots, n.
\]

In order to realize the Cuntz-Krieger algebra using a C*-correspondence, we consider a finite set \( S = \{s_1, \ldots, s_n\} \) and the commutative finite dimensional C*-algebra \( A = C(S) \). It will suffice to work with the basis elements \( f_k \in C(S) \), denoting \( f_k = \chi_{s_k} \). Let \( E \) be the (isomorphism class of) finitely generated (right) Hilbert \( A \)-module, generated by basis vectors

\[
\{e_{ij} : b_{ij} = 1\}.
\]

We define a left module map \( \phi : A \to \text{End}(E) \) and define the module structure

\[
\begin{align*}
\phi(f_k)e_{ij} &= \delta_{ki}e_{ij} \\
e_{ij}f_k &= \delta_{kj}e_{ij} \\
\langle e_{ij}, e_{i'j'} \rangle_A &= \delta_{ii'}\delta_{jj'}f_{jj'}.
\end{align*}
\]

By putting \( e_i = \sum_j e_{ij} \), we have that \( \{e_i\}_i \) still generates \( E \) as a right module, and

\[
\langle e_i, e_j \rangle_A = \delta_{ij} \sum_kb_{ik}f_k.
\]

Moreover

\[
\phi(f_k)e_{ij} = \delta_{ki}e_{ij} = \theta_{e_k,e_k}(e_{ij}) = e_k\langle e_k, e_{ij} \rangle_A = e_k\delta_{ki}f_j = \delta_{ki}e_{kj},
\]

i.e. \( \phi(f_k) = \theta_{e_k,e_k} \). Using the symbols \( S_{ei} \), and defining \( S_i = S_{ei}, \quad Q_i = S_i^*S_i \) and \( P_i = S_iS_i^* \), we get precisely \( P_iP_j = 0 \) for \( i \neq j \) and \( Q_i = \sum_j b_{ij}P_j \), hence \( \mathcal{O}_E \cong \mathcal{O}_B \).

**Definition 2.8.** A Hilbert C*-bimodule \((X, \lambda)\) over \( A \) is a C*-correspondence over \( A \) which in addition is equipped with a left sided inner product

\[
A\langle \cdot, \cdot \rangle : X \times X \to A
\]

such that

\[
A(\xi, \eta)\zeta = \xi\langle \eta, \zeta \rangle_A, \quad \text{for all } \xi, \eta, \zeta \in X.
\]
We note that we may also express the above relation by \( \lambda(\langle \xi, \eta \rangle) = \theta_{\xi, \eta} \).

A Hilbert \( \mathbb{C}^* \)-bimodule \((X, \lambda)\) over \(A\) is called an \textit{equivalence bimodule} if 
\[
A(\langle X, X \rangle) = \langle X, X \rangle_A = A.
\]

Any identity correspondence \(A\) becomes a Hilbert \( \mathbb{C}^* \)-bimodule by defining the canonical left sided inner product \( A \langle x, y \rangle = xy^* \).

**Definition 2.9.** For Hilbert \( \mathbb{C}^* \)-bimodules \((X, \lambda)\) over \(A\) and \((Y, \mu)\) over \(B\), a semicovariant homomorphism \( \pi = (\pi_A, \pi_X) : (X, \lambda) \longrightarrow (Y, \mu) \) is called a \textit{bimodule homomorphism} if 
\[
B(\langle \pi_X(\xi), \pi_X(\eta) \rangle) = \pi_A(\langle \xi, \eta \rangle), \quad \text{for all } \xi, \eta \in X.
\]

**Proposition 2.10.** Let \((X, \lambda)\) be a Hilbert \( \mathbb{C}^* \)-bimodule over \(A\). There exists a \( \mathbb{C}^* \)-algebra \(A \rtimes_X \mathbb{Z}\) and a nondegenerate bimodule homomorphism \(i : (X, \lambda) \longrightarrow A \rtimes_X \mathbb{Z}\) such that for every bimodule homomorphism 
\[
\pi : (X, \lambda) \longrightarrow B
\]
to any identity correspondence \(B\) (with the canonical left sided inner product), there exists a unique *-homomorphism 
\[
\pi_A \times \pi_X : A \rtimes_X \mathbb{Z} \longrightarrow B
\]
with \(\pi_A = (\pi_A \times \pi_X) \circ i_A\) and \(\pi_X = (\pi_A \times \pi_X) \circ i_X\).

**Proof.** We add another relation to the already existing ones in the proof of Proposition 2.2:
\[
T_\xi T^*_\eta = T_A(\langle \xi, \eta \rangle) \quad \text{for } \xi, \eta \in X.
\]

Then define \(A \rtimes_X \mathbb{Z}\) as the completion of this universal *-algebra with the norm as before. The bimodule homomorphism
\[
i = (i_A, i_X) : (X, \lambda) \longrightarrow A \rtimes_X \mathbb{Z}
\]
is naturally defined by \(i_A(a) = Ta\) and \(i_X(\xi) = T_\xi\), and \(\pi_A \times \pi_X(T_a) = \pi_A(a)\), \(\pi_A \times \pi_X(T_\xi) = \pi_X(\xi)\) defines the *-homomorphism \(\pi_A \times \pi_X\). \(\Box\)

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Example 2.11. Considering a C*-algebra $A$ and a fixed automorphism $\alpha \in \text{Aut}(A)$, one has the C*-dynamical system $(A, \mathbb{Z}, \alpha)$ and the associated crossed product $A \rtimes_\alpha \mathbb{Z}$. We define a Hilbert C*-bimodule $X = A$, endowed with the module operations

$$a \cdot x = \alpha(a)x, \quad x \cdot a = xa,$$

for $a \in A, x \in X$,

and inner products

$$A\langle x, y \rangle = \alpha^{-1}(xy^*), \quad \langle x, y \rangle_A = x^*y.$$

With this module structure, one often denotes the module by $\alpha X$ to emphasize the automorphism in question. We have $A \rtimes_\alpha \mathbb{Z} \cong A \rtimes_X \mathbb{Z}$. To show this, we assume for convenience that $A$ is unital. Let $u \in A \rtimes_\alpha \mathbb{Z}$ be the unitary which implements $\alpha$. The element $i_X(1) \in A \rtimes_X \mathbb{Z}$ is a unitary and satisfies

$$i_X(1)i_A(a) = i_X(a), \quad i_A(\alpha^{-1}(a))i_X(1) = i_X(a),$$

and so we define a homomorphism

$$A \rtimes_\alpha \mathbb{Z} \longrightarrow A \rtimes_X \mathbb{Z},$$

$$a \mapsto i_A(a),$$

$$u \mapsto i_X(1)^*.$$

The inverse of this homomorphism is induced by the bimodule homomorphism

$$\pi = (\pi_A, \pi_X) : X \longrightarrow A \rtimes_\alpha \mathbb{Z},$$

$$\pi_A = id_A,$$

$$\pi_X(a) = u^*a.$$

Remark 2.12. We shall need to expand on the notion of Fock space from Remark 2.3. For a Hilbert C*-bimodule $X$ over $A$, we formally define the adjoint bimodule $X^* = \{\xi^* : \xi \in X\}$, endowed with module operations

$$b \cdot \xi^* \cdot a = (a^b)^*, \quad a, b \in A, \xi \in X,$$

$$A(\xi^*, \eta^*) = \langle \xi, \eta \rangle_A, \quad (\xi^*, \eta^*)_A = A(\xi, \eta), \quad \xi, \eta \in X.$$

We adopt the conventions $X^{\otimes 0} = A$, $X^{\otimes n} = X \otimes_A \cdots \otimes_A X$ (interior tensor product, $n$ times), and $X^{\otimes -n} = (X^*)^{\otimes n}$. The Fock space $\mathcal{F}(X)$ of $X$ is defined as

$$\mathcal{F}(X) = \bigoplus_{n=-\infty}^{\infty} X^{\otimes n}.$$

The Fock representation is then constructed as follows. For $a \in A$ we define the operator $\lambda_A(a)$ on $\mathcal{F}(X)$ by

$$\lambda_A(a)(\xi_n)_n = (a\xi_n)_n,$$

i.e. componentwise $A$-left action.
For $\xi \in X$ define the operator $\lambda_X(\xi)$ on $\mathcal{F}(X)$ as: for $\eta \in X^\otimes n$

$$
\lambda_X(\xi)\eta = \begin{cases} 
\xi \otimes \eta & n > 0, \\
\xi \eta & n = 0, \\
A(\xi, \eta^*) & n = -1, \\
A(\xi, \eta_{n+1}) & n < -1, \eta = \eta_1 \otimes \eta_{n+1} \in X^{\otimes -1} \otimes X^\otimes n+1.
\end{cases}
$$

For $\xi \in X^*$ the definition of $\lambda_X(\xi)$ differs slightly,

$$
\lambda_X(\xi)\eta = \begin{cases} 
\xi \otimes \eta & n < 0, \\
\xi \eta & n = 0, \\
A(\xi, \eta^*) & n = 1, \\
A(\xi, \eta_{n-1}) & n > 1, \eta = \eta_1 \otimes \eta_{n-1} \in X \otimes X^\otimes n-1.
\end{cases}
$$

In this notation we get $\lambda_X(\xi)^* = \lambda_X(\xi^*)$. For every $z \in S^1$, $(i_A, z\lambda_X) : (X, A) \rightarrow A \rtimes_X \mathbb{Z}$ is a bimodule homomorphism, which by Proposition 2.10 extends to a $^*$-homomorphism $\gamma_z : A \rtimes_X \mathbb{Z} \rightarrow A \rtimes_X \mathbb{Z}$ satisfying

$$
\gamma_z(i_A(a)) = i_A(a), \text{ and } \gamma_z(i_X(\xi)) = z\lambda_X(\xi).
$$

Then $\gamma : z \mapsto \gamma_z$ is a strongly continuous group of automorphisms, and $\gamma$ is referred to as the dual circle action.

For $z \in S^1$ we have unitaries $U_z \in \mathcal{L}(\mathcal{F}(X))$ which implement the so-called gauge action on $\mathcal{F}(X)$, namely

$$
U_z(\xi_n) = z^n \xi_n, \text{ for } \xi_n \in X^\otimes n,
$$

satisfying the identities

$$
U_z \lambda_A(a) U_z^* = \lambda_A(a),
$$

$$
U_z \lambda_X(\xi) U_z^* = z \lambda_X(\xi),
$$

where $a \in A$, $\xi \in X$, for all $z \in S^1$.

**Remark 2.13.** Faithfulness of the Fock representation may be reasoned as follows: $\lambda_A$ is evidently faithful, for $\lambda_X$ we employ the standard faithful conditional expectations

$$
\Phi : A \rtimes_X \mathbb{Z} \rightarrow A \rtimes_X \mathbb{Z}
$$

$$
\Phi(x) = \int_{S^1} \gamma_z(x) \, dz, \text{ thus } \Phi(A \rtimes_X \mathbb{Z}) = i_A(A),
$$

and

$$
\Psi : \mathcal{L}(\mathcal{F}(X)) \rightarrow \mathcal{L}(\mathcal{F}(X))
$$

$$
\Psi(T) = \int_{S^1} U_z T U_z^* \, dz,
$$

where we integrate using normalized Lebesgue measure. For an element $a \in \ker(\lambda_A \times \lambda_X)$ we get $\lambda_A \times \lambda_X(\Phi(a^*a)) = \Psi(\lambda_A \times \lambda_X(a^*a)) = 0$, and faithfulness of $\lambda_A \times \lambda_X$ on $i_A(A)$ (which by restriction is just $\lambda_A$) implies $a = 0$. 

12
3 Morita equivalence and Takai duality

Working with two Hilbert C*-bimodules, $X$ over a C*-algebra $A$, and $Y$ over a C*-algebra $B$, where the underlying C*-algebras $A$ and $B$ are assumed to be Morita equivalent, we will see that the crossed product C*-algebras $A \rtimes_X Z$ and $B \rtimes_Y Z$ will be Morita equivalent if we are supplied with a certain unitary map. In particular we will achieve that any crossed product over a bimodule will be Morita equivalent to a certain crossed product of the algebra of (generalized) compact operators using a partial automorphism. This will comprise a technical ingredient for the main result of section 5.

We will also address Takai duality. In a more general setting, considering a C*-dynamical system $(A, G, \alpha)$ and the dual system $(A \rtimes \alpha G, \hat{G}, \hat{\alpha})$, it is natural to ask about the structure of $(A \rtimes \alpha G) \rtimes_\hat{\alpha} \hat{G}$, and the Takai duality theorem states that this is isomorphic to $A \otimes K(L^2(G))$. We shall obtain a result analogous to this, when considering the C*-dynamical system $(A \rtimes_X Z, S, \gamma)$ with crossed product $(A \rtimes_X Z) \rtimes_\gamma S^1$.

In the following we shall mean by an $A$—$B$ C*-correspondence a module which is a left Hilbert $A$-module and right Hilbert $B$-module, but with only one right sided $B$-valued inner product. We will first be concerned with extending such an $A$—$B$ C*-correspondence to an $A \rtimes_X Z$—$B \rtimes_Y Z$ C*-correspondence, so that we later may (by using equivalence bimodules instead of merely C*-correspondences) implement the promised Morita equivalence between the crossed products. This is handled in the following pair of results.

**Lemma 3.1.** Let $X$ be a Hilbert C*-bimodule over $A$, and $B$ a C*-algebra. For every pair $(E, v)$ where $E$ is an $A$—$B$ C*-correspondence and

$$v : X \otimes_A E \longrightarrow E \otimes_B B$$

an isometric bimodule map, $E$ extends to an $(A \rtimes_X Z)$—$B$ C*-correspondence denoted $E \times v$.

**Proof.** We define $E \times v = E$ and shall equip it with an $A \rtimes_X Z$ left module action. If we can define a bimodule homomorphism

$$(\lambda_A, \lambda_X) : (A, X) \longrightarrow \mathcal{L}(E),$$

then by Proposition 2.10 it will factor through $A \rtimes_X Z$ and thus yield an $A \rtimes_X Z$ left module action on $E$.

Since $E \otimes_B B \approx E$ canonically, we may write $v : X \otimes_A E \longrightarrow E$. Define $\lambda_A : A \longrightarrow \mathcal{L}(E)$ and $\lambda_X : X \longrightarrow \mathcal{L}(E)$ by

$$\lambda_A(a)\eta = a\eta, \quad \text{for } a \in A, \eta \in E, \quad \text{i.e. usual } A\text{-left action on } E,$$

$$\lambda_X(\xi)\eta = v(\xi \otimes \eta), \quad \text{for } \xi \in X, \eta \in E.$$
These maps satisfy
$$\lambda_X(a\xi b)\eta = v(a\xi b \otimes \eta) = av(\xi \otimes b\eta) = a\lambda_X(\xi)b\eta,$$
in other words $\lambda_X(a\xi b) = \lambda_A(a)\lambda_X(\xi)\lambda_A(b)$, and
$$\langle \lambda_X(\xi_2)^*\lambda_X(\xi_1)\eta, \zeta \rangle_A = \langle v(\xi_1 \otimes \eta), v(\xi_2 \otimes \zeta) \rangle_A = \langle \xi_1 \otimes \eta, \xi_2 \otimes \zeta \rangle_A = \langle \eta, \langle \xi_1, \xi_2 \rangle_A \zeta \rangle_A,$$
so $\lambda_X(\xi_1)^*\lambda_X(\xi_2)\zeta = \langle \xi_1, \xi_2 \rangle_A \zeta$, i.e. $\lambda_X(\xi_1)^*\lambda_X(\xi_2) = \lambda_A(\langle \xi_1, \xi_2 \rangle_A)$. For an element $\zeta = v(\xi_1 \otimes \xi_2)$ we have
$$\langle \lambda_X(\xi), \eta, \zeta \rangle_A = \langle v(\xi \otimes \eta), v(\xi_1 \otimes \xi_2) \rangle_A = \langle \xi \otimes \eta, \xi_1 \otimes \xi_2 \rangle_A = \langle \eta, \langle \xi, \xi_1 \rangle_A \xi_2 \rangle_A$$
which means that $\lambda_X(\xi)^*\zeta = \langle \xi, \xi_1 \rangle_A \xi_2$. Consequently we get
$$\lambda_X(\xi_1)\lambda_X(\xi_2)^*\zeta = \lambda_X(\xi_1)\langle \xi_2, \xi_1 \rangle_A \xi_2 = v(\xi_1 \otimes \langle \xi_2, \xi_1 \rangle_A \xi_2) = v(\xi_1 \otimes (\xi_2, \xi_1)A \xi_2) = v(\xi_1, \xi_2, \xi_1 \otimes \xi_2) = v(\xi_1, \xi_2, \xi_1 \otimes \xi_2) = A(\xi_1, \xi_2)v(\xi_1 \otimes \xi_2) = A(\xi_1, \xi_2)\zeta,$$
i.e. $\lambda_X(\xi_1)\lambda_X(\xi_2)^* = \lambda_A(a(\langle \xi_1, \xi_2 \rangle_A)$, and the bimodule homomorphism properties are thereby verified.

Now we get from Proposition 2.10 the unique *-homomorphism
$$\lambda_A \times \lambda_X : A \times_X Z \longrightarrow \mathcal{L}(E),$$
and we may now define the $A \times_X Z$-left action on $E$ by
$$w \cdot \xi = \lambda_A \times \lambda_X(w)\xi, \quad \text{for } w \in A \times_X Z, \ \xi \in E.$$
\[\square\]

**Corollary 3.2.** Let $E$ be an $A-B$ C*-correspondence with an isometry $v : X \otimes_A E \longrightarrow E \otimes_B Y$. Then $E$ extends to an $A \times_X Z-B \times_Y Z$ C*-correspondence which we denote $\times(E, v)$.

**Proof.** It is evident that $E \otimes_B B \times_Y Z$ is an $A-B \times_Y Z$ C*-correspondence, so if we can produce an appropriate isometric bimodule map, then we may apply the preceding lemma. The isometry we are looking for is indeed
$$v \otimes id_{B \times_Y Z} : X \otimes_A (E \otimes_B B \times_Y Z) \longrightarrow (E \otimes_B B \times_Y Z) \otimes_B B \times_Y Z,$$
where we implicitly understand $Y$ as embedded in $B \times_Y Z$ and thus $E \otimes_B Y$ as embedded in $E \otimes_B B \times_Y Z$. Now by Lemma 3.1, $E \otimes_B B \times_Y Z$ extends to become an $A \times_X Z-B \times_Y Z$ C*-correspondence, and we denote it $\times(E, v)$. \[\square\]
Theorem 3.3. Let $X$ be a Hilbert $C^*$-bimodule over $A$, and $Y$ a Hilbert $C^*$-bimodule over $B$. If $A \sim_M B$ and there exists a unitary

$$u : X \otimes_A E \longrightarrow E \otimes_B Y$$

where $E$ is the $A-B$ equivalence bimodule implementing the Morita equivalence between $A$ and $B$, then $A \rtimes_X Z \sim_M B \rtimes_Y Z$.

Proof. By Corollary 3.2, $(E,u)$ becomes an $A \rtimes_X Z - B \rtimes_Y Z$ equivalence bimodule, and hence implements the Morita equivalence

$$A \rtimes_X Z \sim_M B \rtimes_Y Z.$$

Let $X$ be a Hilbert $C^*$-bimodule over $A$, and define $F_0 = \mathcal{F}(X)(X)_{A}$, $F_1 = \mathcal{F}(X)(X)$. We get the ideals $I = K(F_0)$ and $J = K(F_1)$ in $K(\mathcal{F}(X))$. We have the unitary

$$u : F_0 \otimes_A X \longrightarrow F_1$$

$$(\xi_n)_{n \in \mathbb{Z}} \otimes \xi \mapsto (\xi_{n-1}^1 \otimes \cdots \otimes \xi_{n-1}^{n-1} \otimes \lambda_X(\xi_{n-1}^{n-1}) \xi)_{n \in \mathbb{Z}}$$

where $\lambda_X(\xi_{n-1}^{n-1})$ refers to the Fock operator as defined in Remark 2.12, using the notation $\xi_{m} = \xi_{m}^1 \otimes \cdots \otimes \xi_{m}^m \in X^{\otimes m}$. The mapping $t \mapsto t \otimes 1$ maps $K(F_0)$ onto $K(F_0 \otimes_A X)$ since $(F_0, F_0)_{A} = (A(X, X))$, and we thus get a partial automorphism $\alpha_X$ of $K(\mathcal{F}(X))$,

$$\alpha_X : I \longrightarrow J$$

$$\alpha_X(t) = u(t \otimes 1)u^*.$$ 

The ideal $I$ is a Hilbert $C^*$-bimodule, sometimes denoted $A_X K(\mathcal{F}(X))$, over $K(\mathcal{F}(X))$, where we endow $I$ with the module operations

$$a \cdot \xi = a \xi, \quad \text{usual multiplication between } a \in K(\mathcal{F}(X)) \text{ and } \xi \in I,$$

$$\xi \cdot a = \alpha_X^{-1}(\alpha_X(\xi)a) \quad \text{for } a \in K(\mathcal{F}(X)), \xi \in I,$$

and inner products

$$K(\mathcal{F}(X)) \langle \xi, \eta \rangle = \xi^* \eta, \quad \langle \xi, \eta \rangle_{K(\mathcal{F}(X))} = \alpha_X(\xi^* \eta), \quad \text{for } \xi, \eta \in I.$$ 

Since $\mathcal{F}(X)$ is a full $C^*$-correspondence over $A$, i.e. $(\mathcal{F}(X), \mathcal{F}(X))_A = A$, it is evident that $A \sim_M K(\mathcal{F}(X))$, where $\mathcal{F}(X)$ implements the Morita equivalence. Furthermore $\mathcal{F}(X) \otimes_A X \approx \mathcal{F}(X) \otimes_A (X, X) \otimes_A X \approx F_0 \otimes_A X$, and $I \otimes_{K(\mathcal{F}(X))} \mathcal{F}(X) \approx F_1 \approx F_0 \otimes_A X$. In other words we have a unitary $I \otimes_{K(\mathcal{F}(X))} \mathcal{F}(X) \approx \mathcal{F}(X) \otimes_A X$. Hence we get the following corollary to Theorem 3.3.
Corollary 3.4. For $X$ a Hilbert $C^*$-bimodule over $A$, we have

$$A \rtimes_X Z \sim_M K(\mathcal{F}(X)) \rtimes I \mathbb{Z}.$$ 

Remark 3.5. We may also sometimes write $K(\mathcal{F}(X)) \rtimes_{\alpha_X} Z$ for the above crossed product, emphasizing the partial automorphism.

Remark 3.6. In the case where $X$ is an equivalence bimodule over $A$, so that we have $A(\mathcal{X},X) = (X,X)_A = A$, we get $F_0 = F_1 = \mathcal{F}(X)$ and evidently $I = J = K(\mathcal{F}(X))$, hence $\alpha_X$ becomes an automorphism, $\alpha_X \in \text{Aut}(K(\mathcal{F}(X)))$. One also observes that, upon denoting by $P_n$ the operator of projection onto the $n$-th summand of $\mathcal{F}(X)$, i.e.

$$P_n(\xi_k)_{k \in \mathbb{Z}} = \xi_n \in X^\otimes n,$$

that $\alpha_X$ shifts the projection $P_n$ one step forward to $P_{n+1}$. Indeed, writing $\alpha_X : K(\mathcal{F}(X)) \rightarrow M(K(\mathcal{F}(X))) = \mathcal{L}(\mathcal{F}(X))$ with strictly continuous extension

$$\overline{\alpha}_X : \mathcal{L}(\mathcal{F}(X)) \rightarrow \mathcal{L}(\mathcal{F}(X))$$

we then have, considering an elementary operator $\theta_{\xi,\eta}, \xi,\eta \in \mathcal{F}(X), \zeta = (\zeta_j)_{j \in \mathcal{F}(X)}, \zeta_j = \zeta_1^j \otimes \cdots \otimes \zeta_j^{j-1} \otimes \zeta_0 \in X^\otimes j$, fixed $\zeta_0 \in X$,

$$\overline{\alpha}_X(P_n)(\zeta_j)_j = u(P_n \otimes 1) u^*(\zeta_j)_j$$

$$= u(P_n(\zeta_1^j \otimes \cdots \otimes \zeta_j^{j-1})_j \otimes \zeta_0)$$

$$= \zeta_{n+1}^1 \otimes \zeta_{n+1}^2 \otimes \cdots \otimes \zeta_{n+1}^n \otimes \zeta_0$$

$$= P_{n+1}(\zeta_j)_j.$$

Hence, $\overline{\alpha}_X(P_n) = P_{n+1}$. One also sees that $\overline{\alpha}$ acts as the identity on $\lambda_A(A) \cup \lambda_X(X)$, the image of the Fock representation.

Now for the duality result which will ultimately be used to introduce the Connes spectrum to our setting. We recall the notion of the dual circle action $\gamma : S^1 \rightarrow \text{Aut}(A \rtimes_X Z)$

$$\gamma_z(i_A(a)) = i_A(a), \quad \gamma_z(i_X(\xi)) = z i_X(\xi), \quad \text{for } z \in S^1.$$

In the following result (cf. Takai duality), we consider the $C^*$-dynamical system $(A \rtimes_X Z, S^1, \gamma)$ and determine the dual system $((A \rtimes_X Z) \rtimes_{\gamma} S^1, Z, \hat{\gamma})$.

Theorem 3.7. Let $X$ be an equivalence bimodule over $A$. Then

$$(A \rtimes_X Z) \rtimes_{\gamma} S^1 \cong K(\mathcal{F}(X)).$$
Proof. Consider the C*-dynamical system \((A \rtimes_X \mathbb{Z}, S^1, \gamma)\). The Fock representation \((\lambda_A, \lambda_X)\) from Remark 2.3 induces the *-homomorphism

\[ \lambda_A \times \lambda_X : A \rtimes_X \mathbb{Z} \longrightarrow \mathcal{L}(\mathcal{F}(X)). \]

Denote by \(\lambda_{S^1} : S^1 \longrightarrow \mathcal{L}(\mathcal{F}(X))\) the gauge action on the Fock module, that is, \(\lambda_{S^1}(z) = U_z\) where \(U_z(\xi^n) = z^n \xi^n\), for \(\xi^n \in X^{\otimes n}\). It follows that \((\lambda_A \times \lambda_X, \lambda_{S^1})\) is a covariant representation of the C*-dynamical system \((A \rtimes_X \mathbb{Z}, S^1, \gamma)\) on \(\mathcal{F}(X)\), that is

\[ \lambda_A \times \lambda_X(\gamma_z(i_A(a))) = \lambda_{S^1}(z)\lambda_A \times \lambda_X(i_A(a))\lambda_{S^1}(z)^* \]

\[ \lambda_A \times \lambda_X(\gamma_z(i_X(\eta))) = \lambda_{S^1}(z)\lambda_A \times \lambda_X(i_X(\eta))\lambda_{S^1}(z)^*. \]

Using canonical embeddings \(\sigma : A \rtimes_X \mathbb{Z} \hookrightarrow M((A \rtimes_X \mathbb{Z}) \rtimes_\gamma S^1)\) and \(S^1 \hookrightarrow UM((A \rtimes_X \mathbb{Z}) \rtimes_\gamma S^1)\), the latter embedding being denoted \(z \mapsto u_z\), we know from the general theory that elements \(\sigma(x) \int_{S^1} f(z) u_z dz\), for \(f \in L^1(S^1)\), span a dense subspace of \((A \rtimes_X \mathbb{Z}) \rtimes_\gamma S^1\). It suffices to use the functions \(f(z) = z^n, n \in \mathbb{Z}\), and interpreting \(\xi^n \in X^{\otimes n} \subseteq A \rtimes_X \mathbb{Z}\) as embedded \(\xi^n \hookrightarrow M((A \rtimes_X \mathbb{Z}) \rtimes_\gamma S^1)\), we can then define \(p_n = \int_{S^1} z^{-n} u_z dz\), and say that elements \(\xi^n p_n, m, n \in \mathbb{Z}\), span a dense subspace of \((A \rtimes_X \mathbb{Z}) \rtimes_\gamma S^1\).

Since \((\lambda_A \times \lambda_X, \lambda_{S^1})\) was a covariant representation of \((A \rtimes_X \mathbb{Z}, S^1, \gamma)\) we take the integrated form of it

\[ \Psi = \lambda_A \times \lambda_X \times \lambda_{S^1} : (A \rtimes_X \mathbb{Z}) \rtimes_\gamma S^1 \longrightarrow \mathcal{L}(\mathcal{F}(X)), \]

which acts on the generators by

\[ \Psi(\xi^n p_n) = \lambda_A \times \lambda_X(\xi^n) \int z^{-n} U_z dz = T_{\xi^n} \int z^{-n} U_z dz \]

where by \(T_{\xi^n}\) is meant the operator \(T_{\xi^n} \eta = \xi^n \otimes \eta\), i.e. the usual operator in the Fock representation. Upon writing \(P_n = \int z^{-n} U_z dz\), we have \(\Psi(\xi^n p_n) = T_{\xi^n} P_n \in \mathcal{L}(\mathcal{F}(X))\). We note that \(P_n\) is the projection onto the \(n\)-th summand of \(\mathcal{F}(X)\). Indeed,

\[ \langle P_n(\zeta_k), (\eta_k)_k A = \int z^n \langle U_z(\zeta_k)_k, (\eta_k)_k A dz = \int z^n \sum_k \langle \zeta_k, \eta_k A dz = \langle \zeta_n, \eta_n A, \]

hence \(P_n(\zeta_k) = \zeta_n\), and so we canonically identify \(X^{\otimes n} = P_n \mathcal{F}(X)\).

We now show that the elements \(\Psi(\xi^n p_n)\) are limits of compact operators. Choose \(\{\eta^n_\alpha\}_\alpha \subseteq X^{\otimes n}\) such that \(\{\sum_i A(\eta^n_\alpha, \eta^n_\beta)\}_\alpha\) is an approximate unit in \(A\). For \(\zeta \in \mathcal{F}(X)\) we now get

\[ \sum_i \theta_{\xi^n \eta^n_\alpha, \eta^n_\beta}(\zeta) = \sum_i \xi^n \eta^n_\alpha \langle \eta^n_\alpha, \zeta A = \xi^n \sum_i A(\eta^n_\alpha, \eta^n_\beta) P_\alpha \zeta = \xi^n P_n \zeta \]

\[ = T_{\xi^n} P_n \zeta, \]

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hence \( \Psi(\xi^n p_n) = T_{\xi^n} P_n \in K(\mathcal{F}(X)) \). Furthermore,

\[
\Psi(\xi^n (\eta^n)^* p_n) = T_{\xi^n} T_{\eta^n} P_n = \theta_{\xi^n, \eta^n},
\]

for \( \xi^n \in X^{\otimes m}, \eta^n \in X^{\otimes n}, m, n \in \mathbb{Z} \), which establishes the assertion that \( \Psi((A \rtimes_X \mathbb{Z}) \rtimes_\gamma S^1) = K(\mathcal{F}(X)) \). We now use the faithful left regular representation \( \pi' : (A \rtimes_X \mathbb{Z}) \rtimes_\gamma S^1 \to \mathcal{L}(L^2(S^1, A \rtimes_X \mathbb{Z})) \) which is given by

\[
(\pi'(x)f)(w) = \gamma_w(x)f(w), \quad x \in A \rtimes_X \mathbb{Z}
\]

\[
(\pi'(u_z)f)(w) = f(z^{-1}w), \quad f \in L^2(S^1, A \rtimes_X \mathbb{Z}), w \in S^1, z \in S^1
\]

It is more convenient to consider this representation on \( L^2(\mathbb{Z}) \otimes A \rtimes_X \mathbb{Z} \), so to this end we use the unitary isomorphism (Plancherel theorem)

\[
F : L^2(S^1) \to L^2(\mathbb{Z}).
\]

Since \( L^2(S^1, A \rtimes_X \mathbb{Z}) \cong L^2(S^1) \otimes A \rtimes_X \mathbb{Z} \), we may reformulate the representation \( \pi' \) accordingly, and then put

\[
\pi(\xi^n) = (F \otimes \text{id}) \pi'(\xi^n)(F^* \otimes \text{id}), \quad \pi(u_z) = (F \otimes \text{id}) \pi'(u_z)(F^* \otimes \text{id}),
\]

thus getting the representation

\[
\pi : (A \rtimes_X \mathbb{Z}) \rtimes_\gamma S^1 \to \mathcal{L}(L^2(\mathbb{Z}) \otimes A \rtimes_X \mathbb{Z}).
\]

It follows that for the canonical orthonormal basis \( \{e_n\}_n \) in \( L^2(\mathbb{Z}) \), and any \( x \in A \rtimes_X \mathbb{Z}, \) we have \( F^*(e_n) = g \), where \( g(z) = z^n \). Thus for \( \xi^n \in A \rtimes_X \mathbb{Z}, m \in \mathbb{Z} \), we have

\[
\pi'(\xi^n)(g \otimes x)(w) = \gamma_w(\xi^n)(g \otimes x)(w) = w^m(g \otimes \xi^m x)(w),
\]

and since \( w^m g(w) = w^{m+n} \), we have \( F(w^m g)(w) = e_{m+n} \), so

\[
\pi(\xi^n)(e_n \otimes x) = e_{m+n} \otimes \xi^m x.
\]

Similarly, \( \pi'(u_z)(g \otimes x)(w) = (g \otimes x)(z^{-1}w) \), and since \( g(z^{-1}w) = z^{-n}w^{m+n} \) and \( F \) maps the latter function to \( z^{-n}e_n \), we have \( \pi(u_z)(e_n \otimes x) = z^{-n}e_n \otimes x \).

Hence our representation

\[
\pi : (A \rtimes_X \mathbb{Z}) \rtimes_\gamma S^1 \to \mathcal{L}(L^2(\mathbb{Z}) \otimes A \rtimes_X \mathbb{Z})
\]

is given by

\[
\pi(\xi^n)(e_n \otimes x) = e_{m+n} \otimes \xi^m x, \quad \xi^n \in X^{\otimes m},
\]

\[
\pi(u_z)(e_n \otimes x) = z^{-n}(e_n \otimes x), \quad z \in S^1.
\]
Define a map $V : \mathcal{F}(X) \to l^2(\mathbb{Z}) \otimes A \rtimes X \mathbb{Z}$ by $V(\xi^m) = e_m \otimes \xi^m$. We may now define a homomorphism $V_K(\mathcal{F}(X)) : \mathcal{K}(\mathcal{F}(X)) \to \mathcal{L}(l^2(\mathbb{Z}) \otimes A \rtimes X \mathbb{Z})$ by

$$V_K(\mathcal{F}(X))(\theta_{\xi,\eta}) = \theta_{V(\xi),V(\eta)} \quad \text{for } \xi,\eta \in \mathcal{F}(X).$$

We claim that $V_K(\mathcal{F}(X)) \circ \Psi = \pi$. Indeed, for $n,m \in \mathbb{Z}$, $\zeta = (z_n) \in l^2(\mathbb{Z})$ and $x \in A \rtimes X \mathbb{Z}$, we get

$$V_K(\mathcal{F}(X))(\Psi(\xi^m p_n))(\zeta \otimes x) = V_K(\mathcal{F}(X)) \left( \lim_{\alpha} \sum_{i} \theta_{\xi^m \eta^i_n,\eta^i_n} \right)(\zeta \otimes x)$$

$$= \lim_{\alpha} \sum_{i} \theta_{\xi^m \eta^i_n,\eta^i_n}(\zeta \otimes x) = \lim_{\alpha} \sum_{i} \theta_{\xi^m \eta^i_n,\eta^i_n}(\zeta \otimes x)$$

$$= \lim_{\alpha} \sum_{i} e_{m+n} \otimes \xi^m \eta^i_n(\zeta \otimes x)_{A \rtimes X \mathbb{Z}}$$

$$= \lim_{\alpha} \sum_{i} e_{m+n} \otimes \xi^m \eta^i_n(\zeta \otimes x) = e_{m+n} \otimes \xi^m z_n x$$

$$= \pi(\xi^m p_n)(\zeta \otimes x).$$

The fact that $\pi$ was faithful and $V_K(\mathcal{F}(X)) \circ \Psi = \pi$ implies that $\Psi$ is injective, thus establishing $(A \rtimes X \mathbb{Z}) \rtimes_\gamma S^1 \cong \mathcal{K}(\mathcal{F}(X))$. \hfill \qed

**Remark 3.8.** We wish to identify the system $((A \rtimes X \mathbb{Z}) \rtimes_\gamma S^1, \mathbb{Z}, \hat{\gamma})$ with the system $\mathcal{K}(\mathcal{F}(X)), \mathbb{Z}, \alpha_X)$. We have established the isomorphism

$$\Psi : (A \rtimes X \mathbb{Z}) \rtimes_\gamma S^1 \to \mathcal{K}(\mathcal{F}(X)),$$

so what remains to show is that the isomorphism also intertwines the respective actions, i.e.

$$\Psi \circ \hat{\gamma} = \alpha_X \circ \Psi.$$

To this end, observe that

$$\hat{\gamma}(\xi^m p_n) = \xi^m \int z^n z u_z \, dz = \xi^m p_{n+1},$$

$$\Psi(\xi^m p_{n+1}) = T_{\xi^m} P_{n+1},$$

and on the other hand

$$\Psi(\xi^m p_n) = T_{\xi^m} P_n,$$

$$\alpha_X(T_{\xi^m} P_n) = T_{\xi^m} P_{n+1},$$

since $\alpha_X(P_n) = P_{n+1}$ by Remark 3.6.

Thus we conclude $\Psi \circ \hat{\gamma} = \alpha_X \circ \Psi$ and identify

$$((A \rtimes X \mathbb{Z}) \rtimes_\gamma S^1, \mathbb{Z}, \hat{\gamma}) \cong (\mathcal{K}(\mathcal{F}(X)), \mathbb{Z}, \alpha_X).$$
4 The ideal structure of \( A \rtimes_X \mathbb{Z} \)

For an equivalence bimodule \( X \) over \( A \), we shall see that the Connes spectrum provides important information about the relationship between ideals in \( A \) and ideals in \( A \rtimes_X \mathbb{Z} \). We shall use this to establish a characterization of simplicity of \( A \rtimes_X \mathbb{Z} \). In this paper we always consider closed two-sided ideals (additional conditions will be stated explicitly).

For a Hilbert C*-bimodule \( X \) over a C*-algebra \( A \), one may consider the map

\[
\rtimes : (A, X) \longrightarrow A \rtimes_X \mathbb{Z}
\]

to be a functor on the category of Hilbert C*-bimodules. This formalism can be convenient to describe the relationship between the ideal structure of \( A \) and that of \( A \rtimes_X \mathbb{Z} \).

To a bimodule homomorphism \( j : (A, X) \longrightarrow (B, Y) \), the functor \( \rtimes \) assigns the *-homomorphism

\[
\rtimes(j) : A \rtimes_X \mathbb{Z} \longrightarrow B \rtimes_Y \mathbb{Z}
\]

\[
\rtimes(j)(i_A(a)) = i_B(j_A(a)), \quad \rtimes(j)(i_X(\xi)) = i_Y(j_X(\xi)).
\]

**Definition 4.1.** A sequence of bimodule homomorphisms

\[
0 \longrightarrow (A, X) \xrightarrow{j} (B, Y) \xrightarrow{\pi} (C, Z) \longrightarrow 0
\]

is called *exact* if the sequences

\[
0 \longrightarrow A \xrightarrow{j} B \xrightarrow{\pi} C \longrightarrow 0
\]

\[
0 \longrightarrow X \xrightarrow{j} Y \xrightarrow{\pi} Z \longrightarrow 0
\]

are exact.

**Definition 4.2.** An ideal \( I \subseteq A \rtimes_X \mathbb{Z} \) is called *gauge invariant* if \( \gamma_z(I) \subseteq I \) for all \( z \in S^1 \).

**Proposition 4.3.** Given an exact sequence

\[
0 \longrightarrow (A, X) \xrightarrow{j} (B, Y) \xrightarrow{\pi} (C, Z) \longrightarrow 0
\]

then \( \rtimes(j)(A \rtimes_X \mathbb{Z}) \) is a gauge invariant ideal in \( B \rtimes_Y \mathbb{Z} \).

**Proof.** The image \( \rtimes(j)(A \rtimes_X \mathbb{Z}) \) is generated by \( i_B(j_A(A)) \cup i_Y(j_X(X)) \), hence gauge invariance follows. Furthermore, we have

\[
i_B(\text{im } j_A) \cup i_Y(\text{im } j_X) = i_B(\ker \pi_B) \cup i_Y(\ker \pi_Y),
\]

from which one deduces that \( \rtimes(j)(A \rtimes_X \mathbb{Z}) \) indeed is an ideal. \qed
Proposition 4.4. An exact sequence

\[ 0 \longrightarrow (A, X) \xrightarrow{j} (B, Y) \xrightarrow{\pi} (C, Z) \longrightarrow 0 \]

induces an exact sequence

\[ 0 \longrightarrow A \times_X \mathbb{Z} \xrightarrow{\times(j)} B \times_Y \mathbb{Z} \xrightarrow{\times(\pi)} C \times_Z \mathbb{Z} \longrightarrow 0 \]

Proof. We may denote by \( \gamma^A_z \) and \( \gamma^B_z \) the respective dual circle actions on \( A \times_X \mathbb{Z} \) and \( B \times_Y \mathbb{Z} \), and by \( \Phi_A \) and \( \Phi_B \) the respective faithful conditional expectations (cf. Remark 2.13). The sequence commutes with the respective dual circle actions, hence \( \Phi_B \circ \times(j) = \times(j) \circ \Phi_A \). Let \( a \in \ker \times(j) \). Then \( 0 = \Phi_B(\times(j)(a^*a)) = \times(j)(\Phi_A(a^*a)) \), and since \( \times(j) \) clearly is injective on \( \Phi_A(A \times_X \mathbb{Z}) = i_A(A) \), it follows that \( a = 0 \).

For surjectivity of \( \times(\pi) \), notice that \( \im \times(\pi) = i_C(\pi_B(\mathbb{Z})) \cup i_Z(\pi_Y(\mathbb{Z})) = i_C(\mathbb{Z}) \cup i_Z(\mathbb{Z}) \), and the latter generates \( C \times_Y \mathbb{Z} \). Considering \( \times(\pi) \) modulo the image of \( \times(j) \), we express the quotient map

\[ q : B \times_Y \mathbb{Z} / \im \times(j) \longrightarrow C \times_Z \mathbb{Z}. \]

Since \( \im \times(j) \) is a gauge invariant ideal in \( B \times_Y \mathbb{Z} \), the quotient comes equipped with a faithful conditional expectation which we can denote \( \Phi_q \). Let \( a \in \ker q \). Then \( 0 = \Phi_C(q(a^*a)) = q(\Phi_q(a^*a)) \), but \( \Phi_q(a^*a) \) is an element of \( i_B(B)/j_A(A) \), and

\[ \im \times(j) \cap i_B(B) = i_B(\im j_A) = i_B(\ker \pi_B) \]

implies that \( q \) clearly is faithful on \( \Phi_q(B \times_Y \mathbb{Z}/\im \times(j)) = i_B(B)/j_A(A) \), hence \( a = 0 \). Thus \( q \) is also injective and we conclude that \( \ker \times(\pi) = \im \times(j) \).

Definition 4.5. For a Hilbert C*-bimodule \( X \) over a C*-algebra \( A \), an ideal \( J \subseteq A \) is called \( X \)-bi-invariant if \( A(XJ, X) \subseteq J \) and \( (X, JX)_A \subseteq J \).

Next we observe that an \( X \)-bi-invariant ideal naturally gives rise to an exact sequences of bimodule homomorphisms, and by applying the functor \( \times \) we get a gauge invariant ideal in the crossed product. For an \( X \)-bi-invariant ideal \( J \subseteq A \), consider the exact sequences

\[ 0 \longrightarrow J \xrightarrow{id} A \xrightarrow{\pi_A} A/J \longrightarrow 0 \]

\[ 0 \longrightarrow X \otimes_A J \xrightarrow{id_X \otimes id_J} X \otimes_A A \xrightarrow{id_X \otimes \pi_A} X \otimes_A A/J \longrightarrow 0 \]

where the latter sequence can be also be read

\[ 0 \longrightarrow \overline{XJ} \longrightarrow \overline{XA} \longrightarrow \overline{X(A/J)} \longrightarrow 0 \]
by using the canonical unitary equivalences, and by convention writing $X(A/J)$ for $X \otimes_A A/J$. By Proposition 4.4 we get the exact sequence

$$0 \longrightarrow J \times X Z \overset{x}{\longrightarrow} A \times X Z \overset{x}{\longrightarrow} A/J \times X(A/J) Z \longrightarrow 0$$

It follows from Proposition 4.3 that $J \times X Z = \langle \text{id} \rangle (J \times X Z) \subseteq A \times X Z$ is a gauge invariant ideal. And we can also establish a converse: that any gauge invariant ideal in $A \times X Z$ is precisely of this form. The next result formalizes this discussion.

**Proposition 4.6.** There is a bijection between $X$-bi-invariant ideals in $A$ and gauge invariant ideals in $A \times X Z$, by mapping $J \mapsto J \times X Z$ for an $X$-bi-invariant ideal $J \subseteq A$.

**Proof.** Injectivity of the mapping is immediate by using the faithful conditional expectation $\Phi$ from Remark 2.13, which yields $\Phi(J \times X Z) = i_A(J)$. If $I \subseteq A \times X Z$ is a gauge invariant ideal, put $J = i_A^{-1}(I)$. This is an $X$-bi-invariant ideal in $A$. Indeed, we know that $A \times X Z$ extends the module multiplications and inner products of $(A,X)$, the latter being universally embedded in the former, so that we have

$$i_A : A(XJ,X) \mapsto i_X(X)i_A(J)i_X(X)^* = i_X(X)i_X(X)^* \subseteq I$$

and

$$i_A : \langle X,JX \rangle A \mapsto i_X(X)^*i_A(J)i_X(X) = i_X(X)^*i_X(X) \subseteq I.$$ 

Hence $i_A(\langle X,JX \rangle) \subseteq I$ and $i_A(\langle X,JX \rangle A) \subseteq I$, from which

$$A\langle X,JX \rangle \subseteq i_A^{-1}(I) = J,$$

and denoting by $\Phi_1$ and $\Phi_2$ the respective faithful conditional expectations, we get for any $a \in \ker \pi$ that $0 = \Phi_2(\pi(a)) = \pi(\Phi_1(a))$, but $\Phi_1(a) \in A/J$, hence we must have $a = 0$, so $\pi$ is faithful and thus $I = J \times X Z$. 

Regarding the Connes spectrum (cf. [6]) we need merely to recall that for a $C^*$-dynamical system $(A,G,\alpha)$ the Connes spectrum $\Gamma(\alpha)$ is a certain closed subgroup of the dual group $\hat{G}$. Relevant results regarding the Connes spectrum and the ideal structure of the crossed product will be quoted.
Definition 4.7. For an equivalence bimodule $X$ over a C*-algebra $A$, we define $\Gamma(X) = \Gamma(\alpha_X)$, i.e. the Connes spectrum associated to the C*-dynamical system $(K(F(X)), \mathbb{Z}, \alpha_X)$.

We shall need to cite the following result on the Connes spectrum of the dual system of a C*-dynamical system.

Lemma 4.8. (8.11.7, [6])

Let $(A,G,\alpha)$ be a C*-dynamical system and $(A \rtimes_\alpha G,\Gamma,\hat{\alpha})$ its dual system. An element $t \in G$ belongs to the Connes spectrum $G(\hat{\alpha})$ of the dual system if and only if $I \cap \alpha_t(I) \neq \{0\}$ for every non-zero closed ideal $I \subseteq A$.

In our setting, this result is carried over as follows.

Lemma 4.9. Let $X$ be an equivalence bimodule over $A$. For $z \in S^1$, we have $z \in \Gamma(X)$ if and only if $I \cap \gamma_z(I) \neq \{0\}$ for every closed ideal $I \subseteq A \rtimes_X \mathbb{Z}$.

Proof. Consider the C*-dynamical system $(A \rtimes_X \mathbb{Z}, S^1, \gamma)$ with dual system $((A \rtimes_X \mathbb{Z}) \rtimes_\gamma S^1, \mathbb{Z}, \alpha_X) \cong (K(F(X)), \mathbb{Z}, \alpha_X)$, by Theorem 3.7. The result follows immediately from Lemma 4.8 cited above. \hfill \square

We shall also need to cite the following result.

Lemma 4.10. (Lemma 2.1, [4])

Let $(A,G,\alpha)$ be a C*-dynamical system and assume that for each closed nonzero ideal $I \subseteq A$ and each $t \in G$ we have $I \cap \gamma_z(I) \neq \{0\}$. There is then for each closed nonzero ideal $J$ of $A$ and each compact subset $E$ of $G$ a nonzero element $x \in J$ such that $\alpha_t(x) \in J$ for all $t \in E$.

The preceding result allows us to characterize fullness of the Connes spectrum in terms of ideals.

Theorem 4.11. Let $X$ be an equivalence bimodule over $A$. The following statements are equivalent:

(i) every nonzero closed ideal in $A \rtimes_X \mathbb{Z}$ contains a nonzero gauge invariant ideal,

(ii) $\Gamma(X) = S^1$.

Proof. (i) $\implies$ (ii). $\Gamma(X) \subseteq S^1$ follows by definition, for the reverse inclusion: let $z \in S^1$ and assume $I \subseteq A \rtimes_X \mathbb{Z}$ is a closed ideal. Then by assumption $I$ contains a gauge invariant ideal $J \subseteq I$, i.e. $\gamma_z(J) \subseteq J$. Then obviously $I \cap \gamma_z(I) \supseteq \gamma_z(J) \neq \{0\}$, hence $z \in \Gamma(X)$ by Lemma 4.9.

(ii) $\implies$ (i). $\Gamma(X) = S^1$ means that for all $z \in S^1$ we have $I \cap \gamma_z(I) \neq \{0\}$.
by Lemma 4.9, thus Lemma 4.10 is applicable. For a closed nonzero ideal \( J \subseteq A \rtimes X \), use \( E = S^1 \) in Lemma 4.10 and get \( x \in J \) such that \( \gamma_z(x) \in J \) for all \( z \in S^1 \). Denote by \( J_0 \) the ideal generated by \( x \) and \( \{ \gamma_z(x) \}_{z \in S^1} \) in \( J \). Then \( J_0 \) is a gauge invariant ideal in \( J \).

In the following cited result \( Sp(\alpha) \) refers to the Arveson spectrum (a certain closed subset of the dual of the group in question, cf. [6]).

**Lemma 4.12.** *(Theorem 4.5, [5])*

If \((A,G,\alpha)\) is a C*-dynamical system such that \( A \) is \( G \)-simple, \( \Gamma(\alpha) \) is discrete and \( Sp(\alpha) / \Gamma(\alpha) \) is compact, then \( \Gamma(\alpha)^\perp \) is precisely the subgroup of elements \( t \in G \) such that \( \alpha_t = Ad u \) for some unitary \( G \)-fixpoint \( u \in M(A) \).

Before we start on our main result about simplicity of crossed products over bimodules, we need to undertake a short technical discussion regarding periodicity. Working with a unitary \( u : X^{\otimes n} \to A \) for some \( n \), it turns out that we would benefit from \( u \) satisfying the additional requirement

\[
u(x_0 \otimes \cdots \otimes x_n) = x_0 u(x_1 \otimes \cdots \otimes x_n)\]

for all \( x_0, \ldots, x_n \in X \). The following discussion, which culminates in Lemma 4.13, shows that this additional requirement is safe to adopt after passing to \( n^2 \).

Suppose we have a unitary equivalence \( \beta_1 : X^{\otimes n} \to A \). Using \( \beta_1 \) we then get two unitary equivalences \( X^{\otimes n+1} \to X \) depending on whether we apply \( \beta_1 \) to the first \( n \) or last \( n \) components, i.e.

\[
x_0 \otimes x_1 \otimes \cdots \otimes x_n \mapsto x_0 \beta_1(x_1 \otimes \cdots \otimes x_n), \text{ or} \\
x_0 \otimes x_1 \otimes \cdots \otimes x_n \mapsto \beta_1(x_0 \otimes \cdots \otimes x_{n-1})x_n
\]

for \( x_i \in X, i = 0, \ldots, n \). We know there exists a unitary \( z_1 \in Z(M(A)) \) such that

\[
x_0 \beta_1(x_1 \otimes \cdots \otimes x_n) = z_1 \beta_1(x_0 \otimes \cdots \otimes x_{n-1})x_n.
\]

Define \( \beta_2 = z_1 \beta_1 : X^{\otimes n} \to A \). Re-applying the above argument to \( \beta_2 \) we get a unitary \( z_2 \), and again setting \( \beta_3 = z_2 \beta_2 \), we continue in this manner till we get \( \beta_n \).

For \( x_1, \ldots, x_n, y_1, \ldots, y_n \in X \), we have

\[
\beta_1(x_1 \otimes \cdots \otimes x_{n-1} \otimes x_n \beta_1(y_1 \otimes \cdots \otimes y_n)) = \beta_1(x_1 \otimes \cdots \otimes x_n) \beta_1(y_1 \otimes \cdots \otimes y_n).
\]

On the other hand, by successively applying the relations as in (8), we get
\[
\beta_1(x_1 \otimes \cdots \otimes x_{n-1} \otimes x_n \beta_1(y_1 \otimes \cdots \otimes y_n)) \\
= \beta_1(x_1 \otimes \cdots \otimes x_{n-1} \otimes z_1 \beta_1(x_n \otimes y_1 \otimes \cdots \otimes y_{n-1})y_n) \\
= \beta_1(x_1 \otimes \cdots \otimes x_{n-2} \otimes x_{n-1} \otimes \beta_2(x_n \otimes y_1 \otimes \cdots \otimes y_{n-1})y_n) \\
= \beta_1(x_1 \otimes \cdots \otimes x_{n-1} \beta_2(x_n \otimes y_1 \otimes \cdots \otimes y_{n-1} \otimes y_n) \\
= \beta_1(x_1 \otimes \cdots \otimes x_{n-2} \otimes z_2 \beta_2(x_{n-1} \otimes x_n \otimes y_1 \otimes \cdots \otimes y_{n-2})y_{n-1} \otimes y_n),
\]

continuing in this manner we finally arrive at
\[
= \beta_1(x_1 \beta_n(x_2 \otimes \cdots \otimes x_n \otimes y_1) \otimes y_2 \otimes \cdots \otimes y_{n-1} \otimes y_n) \\
= \beta_1(z_n \beta_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n)y_1 \otimes y_2 \otimes \cdots \otimes y_{n-1}) \\
= \beta_1(z_n \cdots z_1 \beta_1(x_1 \otimes \cdots \otimes x_n)y_1 \otimes \cdots \otimes y_{n-1}) \\
= z_n \cdots z_1 \beta_1(x_1 \otimes \cdots \otimes x_n)\beta_1(y_1 \otimes \cdots \otimes y_{n-1})
\]

and since, to begin with, this was equal to \(\beta_1(x_1 \otimes \cdots \otimes x_n)\beta_1(y_1 \otimes \cdots \otimes y_n)\), we conclude that \(z_n \cdots z_1 = 1\).

**Lemma 4.13.** Defining \(\beta : X^\otimes n^2 \longrightarrow A\) by
\[
\beta(x_1 \otimes \cdots \otimes x_{n^2}) = \beta_1(x_1 \otimes \cdots \otimes x_n) \cdots \beta_n(x_{n^2-n+1} \otimes \cdots \otimes x_{n^2})
\]
we have that \(x_0 \beta(x_1 \otimes \cdots \otimes x_{n^2}) = \beta(x_0 \otimes \cdots \otimes x_{n^2-1})x_{n^2}\).

**Proof.** This is a matter of computation:
\[
x_0 \beta(x_1 \otimes \cdots \otimes x_{n^2}) = x_0 \beta_1(x_1 \otimes \cdots \otimes x_n) \cdots \beta_n(x_{n^2-n+1} \otimes \cdots \otimes x_{n^2}) \\
= z_1 \beta_1(x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1})x_n \beta_2(x_{n+1} \otimes \cdots \otimes x_{2n}) \\
\cdots \beta_n(x_{n^2-n+1} \otimes \cdots \otimes x_{n^2}) \\
= z_1 \beta_1(x_0 \otimes \cdots \otimes x_{n-1})z_2 \beta_2(x_n \otimes x_{n+1} \otimes \cdots \otimes x_{2n-1}) \\
\cdots \beta_n(x_{n^2-n+1} \otimes \cdots \otimes x_{n^2}) \\
continuing in this way, we finally arrive at
\[
= z_1 \beta_1(x_0 \otimes \cdots \otimes x_{n-1})z_2 \beta_2(x_n \otimes \cdots \otimes x_{2n-1})z_3 \beta_3(x_{2n} \otimes \cdots \otimes x_{3n-1})z_4 \\
\cdots \beta_n(x_{n^2-n} \otimes \cdots \otimes x_{n^2-1})x_{n^2} \\
= z_1 z_2 \cdots z_n \beta_1(x_0 \otimes \cdots \otimes x_{n-1})\beta_2(x_n \otimes \cdots \otimes x_{2n-1})\beta_3(x_{2n} \otimes \cdots \otimes x_{3n-1}) \\
\cdots \beta_n(x_{n^2-n} \otimes \cdots \otimes x_{n^2-1})x_{n^2} \\
= \beta(x_0 \otimes \cdots \otimes x_{n^2-1})x_{n^2}.\]

\(\square\)

**Definition 4.14.** We call a Hilbert C*-bimodule \(X\) nonperiodic if \(X^\otimes n \cong A\) implies \(n = 0\).
Example 4.15. Continuing the situation of Example 2.11, we elaborate on what nonexistence of bi-invariant ideals and nonperiodicity amount to for the bimodule $X = \alpha X$ of that example. If $J \subseteq A$ were an $X$-bi-invariant ideal, then
\[ A\langle xa, y \rangle = \alpha^{-1}(xay^*) \in J, \quad \text{for all } x, y \in X, a \in J, \]
\[ \langle x, ay \rangle_A = x^* \alpha(a)y \in J, \quad \text{for all } x, y \in X, a \in J, \]
which is equivalent to $\alpha(J) = J$. Regarding nonperiodicity, we first note that we may identify
\[ X \otimes_m \longrightarrow \alpha_m X \]
\[ x_1 \otimes \cdots \otimes x_m \longrightarrow \alpha^{m-1}(x_1)\alpha^{m-2}(x_2) \cdots \alpha(x_{m-1})x_m. \]
Now if $X$ were to be periodic, say by a unitary equivalence $U': X \otimes^n \longrightarrow A$
then in other words we would have a unitary
\[ U: \alpha^n X \longrightarrow A. \]
In the bimodule $\alpha^n X$ the left module multiplication is $a \cdot y = \alpha^n(a)y$, for $a \in A$ and $y \in \alpha^n X$. Since $U$ is a unitary bimodule map it satisfies $U(a \cdot y) = aU(y)$, which yields
\[ U(\alpha^n(a)y) = aU(y) \]
\[ \alpha^n(a)y = U^*aU(y) \]
i.e. $\alpha^n(a) = U^*aU$, so $\alpha^n$ is an inner automorphism. Hence, in this situation we have
A has no $X$-bi-invariant ideals if and only if $A$ has no $\alpha$-invariant ideals. $X$ is nonperiodic if and only if $\alpha^n$ is not inner for any $n$.

We now give a characterization of simplicity of a crossed product over an equivalence bimodule.

**Theorem 4.16.** Let $X$ be an equivalence bimodule over $A$. The crossed product $A \rtimes_X \mathbb{Z}$ is simple if and only if $A$ contains no $X$-bi-invariant ideals and $X$ is nonperiodic.

**Proof.** Assume that $A$ contains no $X$-bi-invariant ideals and $X$ is nonperiodic. By bijection there are no gauge invariant ideals in $A \rtimes_X \mathbb{Z}$. By Theorem 4.11 it will be sufficient to show $\Gamma(X) = S^1$ to establish simplicity of $A \rtimes_X \mathbb{Z}$. Assume for contradiction that $\Gamma(X) \neq S^1$, so $\Gamma(X)$ must a finite subgroup.
of $S^1$. Considering the C*-dynamical system $(A \rtimes_X \mathbb{Z}) \rtimes_{\gamma} S^1, \mathbb{Z}, \alpha_X)$ we thus have $\Gamma(X) = \Gamma(\alpha_X)$ discrete, and $Sp(\alpha_X)$ being a closed subset of $S^1$, is compact, hence $Sp(\alpha_X)/\Gamma(\alpha_X)$ is also compact. By Lemma 4.12 we then have $n \in \Gamma(\alpha_X)^{+} \subseteq \mathbb{Z}$ such that $\alpha_X^{k} = \text{Ad} u$, for some $u \in U \mathcal{L}(\mathcal{F}(X))$. Recall from Remark 3.6 that $\pi_X(P_m) = P_{m+1}$ and moreover $\pi_X^{k}(P_m) = P_{m+k}$. It now follows that $uP_0u^{*} = \pi_X^{k}(P_0) = P_n$, and applying this operator to an element $(\zeta_m)_{m \in \mathbb{Z}} \in \mathcal{F}(X)$, we get

$$\pi_X^{k}(P_0)(\zeta_m)_m = uP_0u^{*}(\zeta_m)_m = P_n(\zeta_m)_m = \zeta_n \in X^\otimes n.$$ 

This yields a unitary equivalence $A \approx X^\otimes n$ by mapping

$$P_0u^{*}(\zeta_m)_m \longrightarrow uP_0u^{*}(\zeta_m)_m = \zeta_n,$$

in contradiction to the assumption of nonperiodicity, hence we must have $\Gamma(X) = S^1$ and thus $A \rtimes_X \mathbb{Z}$ must be simple.

Now assume that $A \rtimes_X \mathbb{Z}$ is simple. Since in particular there are no gauge invariant ideals in $A \rtimes_X \mathbb{Z}$, then by bijection there are no $X$-bi-invariant ideals in $A$. Assume to the contrary that $X$ is periodic, i.e. there is a unitary $u : X^\otimes n \approx A$ for some $n \neq 0$. Then we define a bimodule homomorphism $\pi = (\pi_A, \pi_X) : (A, X) \longrightarrow \mathcal{L}(A \oplus X \oplus X^{\otimes 2} \oplus \ldots \oplus X^{\otimes n-1})$ by

$$\pi_X(\xi) = T_\xi, \quad \pi_A(a) = T_a, \quad \text{for } \xi \in X, a \in A,$$

as the usual representation on the Fock space, i.e. $T_\xi(\eta) = \xi \otimes \eta$ and $T_a(\eta) = a\eta$, but where we understand summands $X^{\otimes n+k}$ (in the Fock space) as shifted back to $X^{\otimes k}$ for each $k$ using the unitary $u$. Take an element

$$\sum_{i=0}^{n-1} \eta_i \in A \oplus X \oplus X^{\otimes 2} \oplus \ldots \oplus X^{\otimes n-1}, \quad \eta_i \in X^{\otimes i}.$$

Let $\xi_1, \ldots, \xi_n$ be elements of $X$, and compute the composition $T_{\xi_1}T_{\xi_2}\cdots T_{\xi_n}$ on the element $\sum_{i=0}^{n-1} \eta_i$. Dealing with each term $\eta_i$ one at a time, and successively applying the unitary $u$ whenever we reach $n$-tensors, we see that

$$T_{\xi_1}T_{\xi_2}\cdots T_{\xi_n}(\eta_0) = u(\xi_1 \otimes \cdots \otimes \xi_n \eta_0) \in A$$

$$T_{\xi_1}T_{\xi_2}\cdots T_{\xi_n}(\eta_1) = \xi_1 u(\xi_2 \otimes \cdots \otimes \xi_n \otimes \eta_1) \in X$$

$$\vdots$$

$$T_{\xi_1}T_{\xi_2}\cdots T_{\xi_n}(\eta_{n-1}) = \xi_1 \otimes \cdots \otimes \xi_{n-1} u(\xi_n \otimes \eta_{n-1}) \in X^{\otimes n-1}.$$

We may assume that the unitary $u$ satisfies the extra condition as in Lemma
4.13 (after passing to $n^2$ if need be), hence we get

\[
T_{\xi_1} T_{\xi_2} \cdots T_{\xi_n} \left( \sum_{i=0}^{n-1} \eta_i \right)
= u(\xi_1 \otimes \cdots \otimes \xi_n \otimes \eta_0) + \xi_1 u(\xi_2 \otimes \cdots \otimes \xi_n \otimes \eta_1) + \cdots + \xi_1 \otimes \xi_2 u(\xi_3 \otimes \cdots \otimes \xi_n \otimes \eta_1) + \cdots + u(\xi_1 \otimes \cdots \otimes \xi_n) \eta_{n-1}.
\]

This shows that $T_{\xi_1} \cdots T_{\xi_n} = T_u(\xi_1 \otimes \cdots \otimes \xi_n)$, and setting

\[
s = i_X(\xi_1) \cdots i_X(\xi_n) - i_A(u(\xi_1 \otimes \cdots \otimes \xi_n))
\]

we get $s \in \ker(\pi_A \times \pi_X)$. But on the other hand, $s \neq 0$ in $A \rtimes_X Z$, which contradicts the assumed simplicity of $A \rtimes_X Z$. Thus we must have $n = 0$, i.e. $X$ must be nonperiodic.

5 Simplicity of $O_E$

The initial idea of the proof of the simplicity criterion for $O_E$ is to identify $O_E$ with a certain crossed product $A_E \rtimes_{X_E} Z$ over a bimodule. One will also need to take care of the properties of minimality and nonperiodicity under this identification. We handle these issues (in brief) in the following ancillary results, before reaching the main result in Theorem 5.7.

For each $n \in \mathbb{N}$, define $A_E^{(n)} = i_E(E) \cdots i_E(E) i_E(E)^* \cdots i_E(E)^*$, where we have $n$ factors of $i_E(E)$ and of $i_E(E)^*$, a $C^*$-subalgebra in $A \rtimes_E \mathbb{N}$ isomorphic to $K(E^\otimes_n)$. Collecting all these subalgebras together we get the $C^*$-algebra $A_E = \sum_{n=0}^{\infty} A_E^{(n)}$, where by convention $A_E^{(0)} = A$. We see that $A_E = (A \rtimes_E \mathbb{N})^\gamma$ is the gauge invariant subalgebra in $A \rtimes_E \mathbb{N}$. Let $q : A \rtimes_E \mathbb{N} \longrightarrow O_E$ denote the quotient mapping (cf. proof of Proposition 2.5), and define $A_E = q(A_E) \subseteq O_E$. We also define the module $X_E = q(i_E(E)) A_E$, which becomes a Hilbert $C^*$-bimodule over $A_E$ when equipped with the usual algebra multiplications and inner products

\[
\langle x, y \rangle_{A_E} = x^* y, \quad A_E \langle x, y \rangle = xy^*.
\]

Lemma 5.1. $O_E \cong A_E \rtimes_{X_E} Z$.

Proof. By the definitions of $A_E$ and $X_E$ we have a homomorphism

\[
A_E \rtimes_{X_E} Z \longrightarrow O_E.
\]

The inverse homomorphism is obtained by considering the maps

\[
q \circ i_A : A \longrightarrow A \rtimes_E \mathbb{N} \longrightarrow A_E,
q \circ i_E : E \longrightarrow A \rtimes_E \mathbb{N} \longrightarrow X_E,
\]

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thus getting the homomorphism $\tilde{q} : A \rtimes E N \longrightarrow A_E \rtimes X_E Z$. Since clearly $(\ker \tilde{q} \cap A_E) \supset (\ker q \cap A_E)$ and $\ker q$ is generated by $(\ker q \cap A_E)$, we have $\ker q \subset \ker \tilde{q}$, hence we get induced the desired inverse homomorphism $O_E \longrightarrow A_E \rtimes X_E Z$.

We define a map

$$\pi_n : \sum_{k=0}^{n} A_E^{(k)} \longrightarrow \mathcal{L}(E^{\otimes n})$$

$$\pi_n(a) = a_{|E^{\otimes n}},$$

where we understand $A \rtimes E N \subset \mathcal{L}(F^1(E))$ by the Fock representation, so to be precise one has $\pi_n(a) = \lambda_A \times \lambda_X(a)_{|E^{\otimes n}}$ in the notation of Remark 2.3.

**Proposition 5.2.** If $E$ is full and $\epsilon$ is faithful, then

$$\ker q \cap \left( \sum_{k=0}^{n} A_E^{(k)} \right) = \ker \pi_n.$$ 

**Proof.** We refer to [10] for a complete proof, and instead, here we assume $A$ to be unital and $E$ to be finitely generated for convenience. We can find $\{\xi_i\}_{i=1}^m \subset E$ such that $\sum_i \theta_{\xi_i} \xi_i = 1$. Then we get $1 - \sum_i T_{\xi_i} T_{\xi_i}^* = P_0$ and $P_{n+1} = \sum_i T_{\xi_i} P_n T_{\xi_i}^*$, hence $P_n \in \ker q$ for each $n \in \mathbb{N}$. We see that

$$P_n 1 = \sum_{i=1}^{n} P_{n-1} 1 = \sum_{i=1}^{n} \theta_{\xi_i} \xi_i 1 \in \mathcal{L}(F^1(E)),$$

and more specifically, for $\xi_1, \ldots, \xi_k, \zeta_1, \ldots, \zeta_l \in E$, that

$$P_k T_{\xi_1} \cdots T_{\xi_k} T_{\zeta_1}^* \cdots T_{\zeta_l}^* \theta_{\xi_1} \otimes \cdots \otimes \xi_k \otimes \cdots \otimes \zeta_l.$$ 

Hence $K(F^1(E)) \subset \ker q$. On the other hand we know that $\ker q$ is generated by compact operators, thus $\ker q \subset K(F^1(E))$, and therefore we conclude that $\ker q = K(F^1(E))$. Let $a \in \sum_{l=0}^{n} A_E^{(l)}$, then $\pi_{n+k}(a) = \pi_n(a) \otimes 1 \in \mathcal{L}(E^{\otimes n} \otimes E^{\otimes k})$. As $\epsilon$ is faithful and $E$ is full, we conclude that

$$\ker \pi_{n+k} \cap \sum_{l \leq n} A_E^{(l)} = \ker \pi_n.$$ 

If $T \in K(\oplus_{k \leq N} E^{\otimes k})$ then

$$\|a + T\| \geq \|(a + T)_{|E^{\otimes N+1}}\| = \|\pi_{N+1}(a)\| = \|\pi_n(a)\|, \quad N \geq n.$$ 

Therefore

$$\ker q \cap \left( \sum_{k \leq 0} A_E^{(k)} \right) = \ker \pi_n.$$

\[\square\]
Definition 5.3. For a C*-correspondence \((E, \epsilon)\) over a C*-algebra \(A\), we call \(E\) minimal if \(A\) contains no ideals \(J\) with \(\langle E, JE \rangle_A \subseteq J\). We call \(E\) nonperiodic if \(E^\otimes n \approx A\) implies \(n = 0\).

An ideal which satisfies the property mentioned in the above definition is also sometimes called invariant or \(E\)-invariant.

Lemma 5.4. Let \((E, \epsilon)\) be a C*-correspondence over \(A\), and \(F\) a full Hilbert \(A\)-module. Then

\[
K(F) \longrightarrow \mathcal{L}(F \otimes_A E)
\]

maps \(K(F)\) onto \(K(F \otimes_A E)\) if and only if \(\epsilon(A) = K(E)\).

Proof. Assume \(\epsilon(A) = K(E)\). For \(z, w \in E\), find \(a \in A\) such that \(\epsilon(a) = \theta_{z,w}\). Let \(x, y, u \in F\), and \(v \in E\). One computes that

\[
\theta_{x \otimes z, y \otimes w}(u \otimes v) = (\theta_{x,a} \otimes 1)(u \otimes v)
\]

thus we map \(\theta_{x,a,y} \mapsto \theta_{x,a,y} \otimes 1 = \theta_{x \otimes z, y \otimes w}\), and since the range of this map is closed and \(K(F \otimes_A E)\) is linearly generated by elements of the form \(\theta_{x \otimes z, y \otimes w}\), it follows that \(t \mapsto t \otimes 1\) maps onto as claimed.

Conversely, assume that \(t \mapsto t \otimes 1\) maps \(K(F)\) onto \(K(F \otimes_A E)\). We know that \(a \mapsto a \otimes id_{F \otimes A E}\) maps \(A\) onto \(K(F^* \otimes K(F \otimes_A E)) (F \otimes_A E)\). But

\[
F^* \otimes_{K(F \otimes_A E)} (F \otimes_A E) = F^* \otimes_{K(F)} (F \otimes_A E) = (F^* \otimes_{K(F)} F) \otimes_A E
\]

\[
\approx A \otimes_A E \approx E,
\]

from which the claim follows. \(\square\)

Lemma 5.5. Let \((E, \epsilon)\) be a full C*-correspondence over a unital C*-algebra \(A\), with \(\epsilon\) faithful. Then \(\mathcal{X}_E \approx \mathcal{A}_E\) implies \(E \approx A\).

Proof. Assume we have a unitary \(U : \mathcal{X}_E \longrightarrow \mathcal{A}_E\). Putting \(\alpha = U \circ j_E\), we have \(\alpha(\xi a) = \alpha(\xi) j_A(a)\) and \(\alpha(\xi^* \alpha(\eta)) = j_A(\xi, \eta) A\) for all \(\xi, \eta \in E\) and \(a \in A\). Furthermore \(\alpha(E) \mathcal{A}_E = \mathcal{A}_E\). For any \(\xi, \eta, \zeta \in E\) we have

\[
\alpha(\xi) \alpha(\eta) \alpha(\zeta) = U(j_E(\theta_{\xi, \eta} \zeta))
\]

\[
= U(j_{K(E)}(\theta_{\xi, \eta}) j_E(\zeta)) = j_{K(E)}(\theta_{\xi, \eta}) U(j_E(\zeta))
\]

\[
= j_{K(E)}(\theta_{\xi, \eta}) \alpha(\zeta),
\]

thus \(\alpha(\xi) \alpha(\eta)^* = j_{K(E)}(\theta_{\xi, \eta})\). Inductively we obtain

\[
\alpha(E) \cdots \alpha(E) \alpha(E)^* \cdots \alpha(E)^n = \mathcal{A}_E^{(n)}\],

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from which it follows that

\[
\alpha(E)A_E^{(n)}\alpha(E)^* = A_E^{(n+1)},
\]

\[
\alpha(E)^*A_E^{(n+1)}\alpha(E) = A_E^{(n)}.
\]

Denote by \(J\) the image of \(\cup_{n=1}^{\infty} A_E^{(n)} \subseteq A_E \) in \(A_E\). Then

\[
\alpha(E) = \alpha(E)\alpha(E)^*\alpha(E) = j_{K(E)}(K(E))\alpha(E) \subseteq J,
\]

hence \(j_A(A) = \alpha(E)^*\alpha(E) \subseteq J\), so \(J = A_E\). We also know that \(j_{K(E)}(K(E)) = \alpha(E)\alpha(E)^*\) contains an approximate unit for \(A_E\) since \(j_{K(E)}\) is nondegenerate. Thus \(A_E^{(n)} \subseteq A_E^{(n+1)}\) for each \(n\). We wish to show that this is an equality. The ideal \(\langle E, E \rangle_A\) is dense in \(A\), so there exist finite sets \(\{\xi_i\}_i, \{\eta_i\}_i\) in \(E\) such that \(\sum_i \langle \xi_i, \eta_i \rangle_A = 1\), which implies \(\sum_i \alpha(\xi_i)^*\alpha(\eta_i) = 1\). Choose \(n\) such that there exist \(\{y_i\}_i \subseteq A_E^{(n)}\) with \(\|y_i - \alpha(\eta_i)\| \leq (6\sum \|\alpha(\xi_i)\|)^{-1}\) and \(\|\sum \alpha(\xi_i) y_i\| \leq 2\). Define \(z = \sum \alpha(\xi_i)^* y_i\), then for any \(a \in A_E^{(n)}\) such that \(\|a\| \leq 1\), we have \(zaz^* \in A_E^{(n-1)}\) and

\[
\|zaz^* - a\| \leq \|(z - 1)az^*\| + \|az^* - 1\| \leq \|z - 1\|\|(z)\| + 1
\]

\[
= 3\|\sum \alpha(\xi_i)^* y_i - 1\| = 3\|\sum \alpha(\xi_i)^* (y_i - \alpha(\eta_i))\|
\]

\[
\leq 3\sum \|\alpha(\xi_i)\|\|y_i - \alpha(\eta_i)\| \leq \frac{1}{2}.
\]

But for a proper closed subspace \(F_0\) of a Banach space \(F\) we know that for an arbitrary \(\epsilon > 0\) there exists an element in the unit ball of \(F\) with distance to \(F_0\) greater than \(1 - \epsilon\). Thus \(A_E^{(n-1)}\) cannot be a proper subspace of \(A_E^{(n)}\), and it follows that \(k \mapsto k \otimes id_E\) maps \(K(E^{\otimes n-1})\) onto \(K(E^{\otimes n-1} \otimes_A E)\). Using Lemma 5.4 we conclude that \(\epsilon(A) = K(E)\), thus \(A_E^{(n)} = A_E^{(n+1)}\) for all \(n\). Therefore \(A_E \cong A\), and hence \(E = X_E \cong A_E \cong A\). \(\square\)

Let \((E, e)\) be a full C*-correspondence over \(A\). For an ideal \(J \subseteq A\), one defines a set

\[
E^{-1}(J) = \{a \in A : \langle E, aE \rangle_A \subseteq J\},
\]

and says that the ideal \(J\) is saturated if \(E^{-1}(J) \subseteq J\). For any proper ideal \(J \subseteq A\), we have that \(E^{-1}(J)\) is also a proper ideal. If \(J\) is an ideal such that \(\langle E, JE \rangle_A \subseteq J\), then \(J \subseteq E^{-1}(J)\). Maximal invariant ideals are saturated; if \(J\) is a maximal invariant ideal, then \(E^{-1}(J)\) is invariant and contains \(J\), hence must equal \(J\). If the C*-algebra \(A\) is unital, then the property of minimality is equivalent to nonexistence of nontrivial saturated and invariant ideals. For Hilbert C*-bimodules, saturated invariant ideals are bi-invariant, since assuming \(\langle X, JX \rangle_A \subseteq J\) implies that \(\langle X, A\langle J, X \rangle X \rangle_A = \langle X, XJ\langle X, X \rangle_A \rangle_A = \langle X, X \rangle_AJ\langle X, X \rangle_A \subseteq J\), thus also
\[ A(XJ, X) \subseteq J. \] We also remark that for a C*-correspondence \((E, \varepsilon)\), minimality implies faithfulness of \(\varepsilon\), since \(\ker \varepsilon\) would have been an invariant ideal.

**Lemma 5.6.** Let \((E, \varepsilon)\) be a full C*-correspondence over a unital C*-algebra \(A\), with \(\varepsilon\) faithful. Then \((E, \varepsilon)\) is minimal and nonperiodic if and only if \((\mathcal{X}_E, l_E)\) is minimal and nonperiodic.

**Proof.** Assume that \((\mathcal{X}_E, l_E)\) is periodic, i.e. there is an \(n\) such that \(\mathcal{X}_E^{\otimes n} \approx \mathcal{X}_E^n\). Then by Lemma 5.5 we have that \(E^{\otimes n} \approx A\), i.e. \(E\) is periodic. On the other hand, if \(E^{\otimes n} \approx A\), then \(\mathcal{X}_E^{\otimes n} \approx \mathcal{X}_E^n\). Assume that \((E, \varepsilon)\) is minimal. Let \(J \subseteq \mathcal{A}_E\) be an ideal such that

\[ \langle \mathcal{X}_E, J \mathcal{X}_E \rangle_{\mathcal{A}_E} \subseteq J. \]

Since \(A\) is unital and due to the discussion of saturated ideals, we can assume that \(J\) is \(\mathcal{X}_E\)-bi-invariant. The bimodule homomorphism

\[ (q_{\mathcal{A}_E}, q_{\mathcal{X}_E}) : (X_E, \lambda_E) \rightarrow (\mathcal{X}_E, l_E) \]

gives that \(\tilde{J} = q_{\mathcal{A}_E}^{-1}(J)\) is \(X_E\)-bi-invariant, and in particular that

\[ \langle E^{\otimes n}, \tilde{J} E^{\otimes n} \rangle_A \subseteq \tilde{J} \]

and

\[ i_{K(E^{\otimes n})}(\langle K(E^{\otimes n}), \tilde{J}, E^{\otimes n} \rangle) \subseteq \tilde{J} \]

for each \(n > 0\). We have that \(i_{K(E^{\otimes n})}(K(E^{\otimes n})) \cap \tilde{J} \neq 0\) if and only if

\[ j_{K(E^{\otimes n})}(K(E^{\otimes n})) \cap J \neq 0, \]

moreover \(i_{K(E^{\otimes n})}(K(E^{\otimes n})) \cap J \neq 0\) implies that \(\langle E^{\otimes n}, (i_{K(E^{\otimes n})}(K(E^{\otimes n})) \cap J), E^{\otimes n} \rangle_A \subseteq \tilde{J} \cap A\) is nonzero. Minimality of \((E, \varepsilon)\) now implies that \(\tilde{J} \cap A = A\), from which it follows that

\[ \langle K(E^{\otimes n}), i_{K(E^{\otimes n}}(K(E^{\otimes n})) \cap \tilde{J}, E^{\otimes n} \rangle \subseteq \langle i_{K(E^{\otimes n}}(K(E^{\otimes n})) \cap \tilde{J}, A \rangle \subseteq \langle E^{\otimes n}, \tilde{J}, E^{\otimes n} \rangle \]

since \(\tilde{J}\) was \(X_E\)-bi-invariant. This contradicts that \(\tilde{J}\) was proper, hence

\[ j_{K(E^{\otimes n})}(K(E^{\otimes n})) \cap J = 0 \]

for each \(n\), i.e. \(J \cap \sum_{k \leq n} A_E^{(k)} = 0\) since it is orthogonal to the essential ideal \(j_{K(E^{\otimes n})}(K(E^{\otimes n}))\). Since the span of \(\bigcup_{n=1}^{\infty} A_E^{(n)}\) was dense in \(A_E\), it follows that \(J = 0\).

Conversely, let \(I \subseteq A\) be a nontrivial ideal such that \(\langle E, IE \rangle_A \subseteq I\). One may again assume that \(I\) is saturated. We get a nontrivial ideal

\[ J = \{ x \in A \times_E \mathbb{N} : \langle \mathcal{F}^{1}(E), \lambda_A \times \lambda_X(x) \mathcal{F}^{1}(E) \rangle_A \subseteq I \} \]

in \(A \times_E \mathbb{N}\). This is because \(I \subseteq J\) and also \(J \cap A \subseteq I\). Then \(J\) maps to a nontrivial ideal in \(O_E\) by faithfulness of \(j_A\) (following from that of \(\varepsilon\)).
Theorem 5.7. Let \((E, \epsilon)\) be a full \(C^*\)-correspondence over a unital \(C^*\)-algebra \(A\), with \(\epsilon\) faithful. Then \(\mathcal{O}_E\) is simple if and only if \(E\) is minimal and nonperiodic.

Proof. Assuming \(\mathcal{O}_E \cong A_E \rtimes \mathcal{X}_E \mathbb{Z}\) to be simple, it follows from the second part of the proof of Theorem 4.16 that \(A_E\) has no \(\mathcal{X}_E\)-bi-invariant ideals and \(\mathcal{X}_E\) is nonperiodic regardless of \(\mathcal{X}_E\) being an equivalence bimodule. It then follows from Lemma 5.6 that \(E\) is minimal and nonperiodic.

Conversely, assume that \(E\) is minimal and nonperiodic. Then by Lemma 5.6 \((\mathcal{X}_E, l_E)\) is also minimal and nonperiodic. We have \(\mathcal{O}_E \cong A_E \rtimes \mathcal{X}_E \mathbb{Z}\), but since \((\mathcal{X}_E, l_E)\) may not be an equivalence bimodule in general, Theorem 4.16 is not applicable immediately. So in the following we assume \((\mathcal{X}_E, l_E)\) to not be an equivalence bimodule. By Corollary 3.4 we have

\[ A_E \rtimes \mathcal{X}_E \mathbb{Z} \sim_M B \rtimes \beta \mathbb{Z}, \]

where \(\beta : I \to B\) is a partial automorphism, \(I = K(\mathcal{F}(\mathcal{X}_E), A_E(\mathcal{X}_E, \mathcal{X}_E))\) is an essential ideal in \(B = K(\mathcal{F}(\mathcal{X}_E))\). Compose the extension

\[ \overline{\beta} : M(I) \to B \]

with the restriction map \(M(B) \to M(I)\) to get an injective \(*\)-homomorphism \(\hat{\beta} : M(B) \to M(B), \text{ so } B \hookrightarrow \hat{\beta}(B)\). Then \(\{\hat{\beta}^n(B)\}_{n \in \mathbb{N}}\) is an increasing sequence of ideals

\[ B \hookrightarrow \hat{\beta}(B) \hookrightarrow \hat{\beta}^2(B) \hookrightarrow \hat{\beta}^3(B) \hookrightarrow \ldots \]

and we put \(C = \bigcup_{n=1}^\infty \hat{\beta}^n(B)\). We get an automorphism \(\gamma = \hat{\beta}|_C \in \text{Aut}(C)\). Denote by \(X\) the Hilbert \(C^*\)-bimodule corresponding to \(\beta\), i.e. \(X = B\) and module multiplication and inner products are defined by

\[ a \cdot x \cdot b = \hat{\beta}(a) xb, \quad B(x, y) = \hat{\beta}^{-1}(xy^*), \quad (x, y)_B = x^*y, \quad a, b, x, y \in B. \]

Using the isomorphism \(\beta\) between \(I\) and \(B\) we thus understand \(B \rtimes \beta \mathbb{Z}\) (cf. Corollary 3.4) as \(B \rtimes X \mathbb{Z}\). Minimality of \(\mathcal{X}_E\) means there are no gauge invariant ideals in \(A_E \rtimes \mathcal{X}_E \mathbb{Z}\), and by Morita equivalence there can be no gauge invariant ideals in \(B \rtimes X \mathbb{Z}\), hence \(B\) can have no \(X\)-bi-invariant ideals. We denote by \((Y, \gamma)\) the Hilbert \(C^*\)-bimodule corresponding to \(\gamma\) (with \(Y = C\)), which then is an equivalence bimodule.

Simplicity of \(C \rtimes_Y \mathbb{Z} = C \rtimes \gamma \mathbb{Z}\) is, by Theorem 4.16, equivalent to nonperiodicity of \(Y\) and nonexistence of \(Y\)-bi-invariant ideals in \(C\). If \(Y\) were periodic, say \(Y^\otimes m \approx C\), then \(\gamma^m(B) = B\), hence we would have \(C = B\) so \(Y\) would in fact be an equivalence bimodule, thus also \((\mathcal{X}_E, l_E)\) to begin with, contrary to our working assumption.

Assume that \(J \subseteq C\) is \(Y\)-bi-invariant ideal. Then \(J_0 = B \cap J\) is an ideal in \(B\) with \(\langle X, J_0 X \rangle_B = B^* \hat{\beta}(J_0)B \subseteq B^* \gamma(J)B \subseteq J \cap B\). Likewise we have
\(B(J_0, X) = \tilde{\beta}^{-1}(BJ_0B) \subseteq J \cap B\). Hence \(J_0 = B\) since \(B\) had no \(X\)-bi-invariant ideals, thus \(J = C\).

We now show \(B \rtimes X \mathcal{Z} \cong C \rtimes Y \mathcal{Z}\). Since \(B \hookrightarrow C \subseteq M(B)\), and then in particular \(X \hookrightarrow Y\), the inclusion induces a \(*\)-homomorphism \(\pi : B \rtimes X \mathcal{Z} \longrightarrow C \rtimes Y \mathcal{Z}\).

We also have a bimodule homomorphism \((\psi_C, \psi_Y) : (Y, \gamma) \longrightarrow M(B \rtimes X \mathcal{Z})\) where

\[
\psi_C \text{ is the restriction of } M(B) \hookrightarrow M(B \rtimes X \mathcal{Z}) \text{ to } C,
\]

\[
\psi_Y(y) = v\psi_C(y), \text{ where } v \text{ is the isometry s.t. } \beta(b) = v^*bv.
\]

Hence there exists a unique \(*\)-homomorphism \(\psi : C \rtimes Y \mathcal{Z} \longrightarrow M(B \rtimes X \mathcal{Z})\), and moreover \(\psi \circ \pi = id\), since \(\pi\) was an inclusion in one direction, and \(\psi\) is the restriction in the opposite direction. Hence \(\pi\) is injective. Since \(\psi\) maps \(\pi(B \rtimes X \mathcal{Z})\) onto the ideal \(B \rtimes X \mathcal{Z} \subseteq M(B \rtimes X \mathcal{Z})\), thus \(\pi(B \rtimes X \mathcal{Z})\) is itself an ideal, but by simplicity of \(C \rtimes Y \mathcal{Z}\) we must have \(\pi(B \rtimes X \mathcal{Z}) = C \rtimes Y \mathcal{Z}\), i.e. \(\pi\) is surjective.

\(\square\)

In Example 2.6 the Cuntz algebra \(O_n\) was realized as \(O_H\) using \(H = C^n\).

The module \(H\) is of course minimal since \(C\) is already simple. Nonperiodicity of \(H\) is also immediate since

\[
H^\otimes m = C^n \otimes \cdots \otimes C^n \approx C
\]

is impossible for any \(m \neq 0\) by taking the dimensions into account. Hence the application of the simplicity result is trivial for the classical Cuntz algebra.

6 Cuntz-Pimsner algebras of self-similar group actions

Given a self-similar group action of a group \(G\) on a sequence space \(X^\omega\), one associates a bimodule \(\Phi\) over the group algebra \(CG\), encoding the self-similarity. Considering a certain completion \(A_\Phi\) of \(CG\), one has that \(\Phi\) becomes a \(C^*\)-correspondence over \(A_\Phi\). In this section we aim to apply the simplicity result to the Cuntz-Pimsner algebra \(O_\Phi\).

We begin by recalling the basic terminology and facts regarding self-similar group actions, self-similar completions and self-similarity bimodules from [2].

Let \(X\) be a finite set, and denote by \(X^\omega\) the set of all infinite sequences (words) of the form \(x_1x_2x_3\ldots\), where \(x_i \in X\) for each \(i\). Equip \(X^\omega\) with the direct product (Tikhonov) topology coming from the discrete sets \(X\). The basis of the topology then consists of cylindrical sets \(a_1a_2a_3\ldots a_nX^\omega, a_i \in X\). One denotes by \(X^*\) the set of all finite words \(x_1\ldots x_n\), together
with the empty word, thus \( X^* = \bigcup_{n \geq 0} X^n \). Given a sequence \( w \in X^\omega \) and a finite word \( v \), we understand \( vw \) as the concatenation of the two.

We shall be concerned with a countable group \( G \) acting faithfully on the space \( X^\omega \), writing \( g(w) \) for the action of \( g \in G \) on \( w \in X^\omega \). The group action will be a so-called *self-similar* group action, meaning that for every \( g \in G \) and \( x \in X \) there exist \( h \in G \) and \( y \in X \) such that

\[
g(xw) = yh(w), \quad \text{for all } w \in X^\omega.
\]

One writes the last equation, referred to as the self-similarity condition, formally as

\[
g \cdot x = y \cdot h.
\]

In the following we fix a self-similar and minimal group action of a group \( G \) on \( X^\omega \) for a finite set \( X = \{x_1, \ldots, x_d\} \). Denote by \( \Phi \) the free (right) \( CG \)-module, free basis being \( X \). The module \( \Phi \) has the \( CG \)-valued sesquilinear form

\[
\left\langle \sum_{x \in X} x \cdot a_x, \sum_{x \in X} x \cdot b_x \right\rangle_{CG} = \sum_{x \in X} a^*_x b_x.
\]

We define a left module multiplication using the self-similarity condition, namely for any \( g \in G \) and \( x \in X \), we define the left multiplication by \( g \cdot x = y \cdot h \), where \( h \in G \) and \( y \in X \) are such that \( g(xw) = yh(w) \) for all \( w \in X^\omega \). The multiplication extends by linearity to the whole module \( \Phi \), giving a map from \( G \) to \( \text{End}(\Phi) \). This map then extends to

\[
\phi : CG \longrightarrow \text{End}(\Phi)
\]

which for group elements \( g \in G \) is defined as

\[
\phi(g) \sum_{x \in X} x \cdot a_x = \sum_{x \in X} g \cdot x \cdot a_x = \sum_{x \in X} y_x \cdot h_x a_x, \quad \text{for } g \cdot x = y_x \cdot h_x,
\]

thus supplying \( \Phi \) with the structure of a left \( CG \)-module as well. One refers to \( \Phi \) as the *self-similarity bimodule*, although in our terminology it is more rightfully a \( C^* \)-correspondence (lacking the left sided inner product in order for it to be called a bimodule).

One would like to complete the algebra \( CG \) in such a way that the self-similarity bimodule \( \Phi \) would become a Hilbert bimodule, thus allowing us to speak of the \( C^* \)-correspondence \( (\Phi, \phi) \). Such a completion of \( CG \) is called a *self-similar completion*. Among such completions, there exists a certain unique minimal completion, which is defined using the notion of generic points.

**Definition 6.1.** A point \( w \in X^\omega \) is called \( G \)-generic if for every \( g \in G \) one has either \( g(w) \neq w \), or there exists a neighborhood \( U \) of \( w \), such that every point in \( U \) is fixed by \( g \).
Denoting by \( G(w) \) the \( G \)-orbit of a fixed \( G \)-generic point \( w \in X^\omega \), one introduces the so called permutation representation \( \pi_w \) of \( CG \) on \( l^2(G(w)) \), which is defined, for \( g \in G \), \( f \in l^2(G(w)) \), \( f = \sum_{u \in S} \alpha_u u \), \( \alpha_u \in \mathbb{C} \), \( S \subset G(w) \) a finite subset,

\[
\pi_w(g) f = \sum_{u \in S} \alpha_u g(u)
\]

and then extended to \( CG \) by linearity and continuity. Then we define a norm \( || \cdot ||_w \) on \( CG \) by \( ||a||_w = ||\pi_w(a)|| \) as the operator norm using the representation \( \pi_w \), for \( a \in CG \). The completion of \( (CG, || \cdot ||_w) \) is denoted \( A_\Phi \). It is shown in [2] that \( A_\Phi \) is a self-similar completion of \( CG \) (and not depending on \( w \)), i.e. \( \Phi \) becomes a Hilbert C*-module over \( A_\Phi \), hence allowing us to work with the C*-correspondence \( (\Phi, \phi) \) over \( A_\Phi \).

In order to conclude that \( O_\Phi \) is simple, we can by Theorem 5.7 show that \( \Phi \) is minimal and nonperiodic.

To address minimality, we introduce the following notation for convenience,

\[
S_x : A_\Phi \longrightarrow A_\Phi
\]

\[
S_x(a) = (x \cdot e, a(x \cdot e))_{A_\Phi}
\]

Then invariance of an ideal \( J \subseteq A_\Phi \), namely \( \langle \Phi, J \Phi \rangle_{A_\Phi} \subseteq J \), in particular means \( S_x(J) \subseteq J \) for all \( x \in X \). The operators \( S_x \) are bounded, since by writing \( \xi = x \cdot e \), we have \( ||\xi|| = ||\langle \xi, \xi \rangle_{A_\Phi}||_{1/2} = ||e^* e||_{1/2} = ||e|| = 1 \), and then

\[
||S_x(a)|| = ||\langle \xi, a\xi \rangle_{A_\Phi}|| \leq ||\xi|| \cdot ||a\xi|| \leq ||\xi||^2 ||a|| = ||a||
\]

thus \( ||S_x|| \leq 1 \).

Let \( w \in X^\omega \) be a \( G \)-generic point, meaning that for each \( g \in G \), either \( g(w) \neq w \), or there exists a neighborhood of \( w \) which is fixed by \( g \). In the following presume \( w = x_1 x_2 x_3 \ldots \).

**Proposition 6.2.** Let \( g \in G \). Then \( S_{x_n} \cdots S_{x_1}(g) \neq 0 \) for all \( n \in \mathbb{N} \) if and only if \( g(w) = w \).

**Proof.** \( S_{x_1}(g) \neq 0 \) is equivalent to \( g \cdot x_1 = x_1 \cdot h_1 \), for some \( h_1 \in G \), and then \( S_{x_1}(g) = h_1 \). \( S_{x_2}(h_1) \neq 0 \) is equivalent to \( h_1 \cdot x_2 = x_2 \cdot h_2 \) for some \( h_2 \in G \), and so \( S_{x_2}(h_1) = h_2 \). Continuing in this manner till the \( n \)-th iteration, we get

\[
g(x_1 \ldots x_n \ldots) = x_1 h_1(x_2 \ldots x_n \ldots) = x_1 x_2 h_2(x_3 \ldots x_n \ldots) = \ldots = x_1 \ldots x_n h_n \ldots
\]

It follows that \( S_{x_n} \cdots S_{x_1}(g) \neq 0 \) for all \( n \) if and only if \( g(w) = w \). \( \square \)
It follows from Proposition 6.2 that for an element \( g \in G \), if \( g(w) \neq w \) then there exists \( m \in \mathbb{N} \) such that \( S_{x_m} \cdots S_{x_1}(g) = 0 \). In the situation that \( g(w) = w \), it would be beneficiary to know what happens when successive applications of the maps \( S_x \) are done to \( g \), other than it being non-zero.

**Proposition 6.3.** Let \( g \in G \). If \( g(w) = w \) then there exists \( m \in \mathbb{N} \) such that \( S_{x_m} \cdots S_{x_1}(g) = e \).

**Proof.** The point \( w \in X^w \) was \( G \)-generic, so the assumption \( g(w) = w \) means there exists a neighborhood of \( w \) which is fixed by \( g \). Cylindrical sets constitute a neighborhood basis, thus there exists a cylindrical set \( x_1 \ldots x_mX^w \) containing \( w \) and being kept fixed under \( g \). Writing \( g \cdot x_i = x_i \cdot h_i \), for \( i = 1, \ldots, m \), we get

\[
g(x_1 \ldots x_my_1y_2 \ldots) = x_1h_1(x_2 \ldots x_my_1y_2 \ldots) = \ldots = x_1 \ldots x_mh_m(y_1y_2 \ldots)
\]

for all \( y_1y_2 \ldots \in X^w \). Thus \( h_m(y_1y_2 \ldots) = y_1y_2 \ldots \) for all \( y_1y_2 \ldots \in X^w \), which implies \( h_m = e \) due to the faithfulness of the action. Hence we get \( S_{x_m} \cdots S_{x_1}(g) = e \) as was to be shown. \( \square \)

Having obtained \( A_\Phi \) by representing \( C\mathcal{G} \) on \( l^2(G(w)) \) and completing, we now define a state \( \phi_w : A_\Phi \rightarrow A_\Phi \)

\[
\phi_w(a) = \langle a\delta_w, \delta_w \rangle_{l^2}.
\]

It is clear that for \( b \in C\mathcal{G} \), \( \phi_w \) picks out the coefficients in front of isotropy group elements. More precisely, for \( b = \sum_i b_i g_i \), then \( \phi_w(b) = \sum_i \delta_{g_i} b_i \).

The state \( \phi_w \) may thus be advantegously used in conjunction with the preceding propositions. Also note that for an ideal \( J \subseteq A_\Phi \) one has \( \phi_w|J = 0 \) if and only if \( \pi_w(a) \delta_w = 0 \) for all \( a \in J \), which is equivalent to \( \pi_w(a) \pi_w(b) \delta_w = 0 \) for all \( a \in J, b \in A_\Phi \), which in turn holds if and only if \( \pi_w(a) = 0 \), i.e. \( J \) itself is zero. Hence there do exist elements in \( J \) on which \( \phi_w \) will be nonzero.

**Lemma 6.4.** For every \( b \in C\mathcal{G} \) there exists \( m \in \mathbb{N} \) such that \( S_{x_m} \cdots S_{x_1}(b) = \phi_w(b)e \).

**Proof.** Assume \( b = \sum_{i=1}^N b_i g_i \). Define the sets \( I_1 = \{ i : g_i(w) \neq w \} \) and \( I_2 = \{ j : g_j(w) = w \} \). From Proposition 6.2 it follows that for each \( i \in I_1 \) there exists \( n(i) \in \mathbb{N} \) such that \( S_{x_{n(i)}} \cdots S_{x_1}(g_i) = 0 \), and likewise from Proposition 6.3 it follows that for each \( j \in I_2 \) there exists \( m(j) \in \mathbb{N} \) such that \( S_{x_{m(j)}} \cdots S_{x_1}(g_j) = e \). Put \( m = \max\{ n(i), m(j) : i \in I_1, j \in I_2 \} \). Then for each \( g_i \) we have

\[
S_{x_m} \cdots S_{x_1}(g_i) = \begin{cases} 0 & \text{if } i \in I_1, \\ e & \text{if } i \in I_2. \end{cases}
\]

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Hence

\[ S_{x_m} \cdots S_{x_1}(b) = S_{x_m} \cdots S_{x_1}\left(\sum_{i=1}^{N} b_i g_i\right) = \left(\sum_{i \in I_2} b_i\right) e = \phi_w(b)e. \]

\[ \blacksquare \]

**Proposition 6.5.** \( \Phi \) is minimal.

**Proof.** Let \( \{0\} \neq J \subseteq A_{\Phi} \) be an ideal such that \( \langle \Phi, J\Phi \rangle_{A_{\Phi}} \subseteq J \), hence also \( S_x(J) \subseteq J \) for all \( x \in X \). Let \( a^*a \in J \) be such that \( \phi_w(a^*a) > 0 \). For an arbitrary \( \epsilon > 0 \) find \( b \in \mathbb{C}G \) such that \( ||a^*a - b|| < \epsilon \). Suppose \( b = \sum_{i=1}^{N} b_i g_i \).

Then we also have \( |\phi_w(a^*a) - \phi_w(b)| < \epsilon \). By Lemma 6.4 there exists \( m \in \mathbb{N} \) such that \( S_{x_m} \cdots S_{x_1}(b) = \phi_w(b)e \). The operators \( S_x \) are contractions, so we have \( ||S_{x_m} \cdots S_{x_1}(a^*a) - S_{x_m} \cdots S_{x_1}(b)|| < \epsilon \). It follows that

\[ ||S_{x_m} \cdots S_{x_1}(a^*a) - \phi_w(a^*a)e|| \leq ||S_{x_m} \cdots S_{x_1}(a^*a) - \phi_w(b)e|| + ||\phi_w(b)e - \phi_w(a^*a)e|| = 2\epsilon \]

This shows that the sequence \( \{S_{x_m} \cdots S_{x_1}(a^*a)\}_{m \in \mathbb{N}} \subseteq J \) converges to the scalar multiple \( \phi_w(a^*a)e \), hence \( \phi_w(a^*a)e \in J \) and thus \( J \) must be the whole algebra \( A_{\Phi} \).

\[ \blacksquare \]

We turn to discuss the nonperiodicity of the bimodule \( \Phi \). Let’s assume that we have a unitary \( u : \Phi \to A_{\Phi} \), and denote for shorthand \( u_x = u(x \cdot e) \), for \( x \in X \). These elements clearly satisfy

\[ u_x^* u_x = e, \quad u_x^* u_y = 0 \text{ for } x \neq y, \quad gu_x = u_y h \text{ for } g \cdot x = y \cdot h. \]

For a finite word \( v \in X^\omega \) define the map \( T_v : w \mapsto vw \), with partially defined inverse \( T_v^* : vw \mapsto w \).

**Definition 6.6.** A point \( w \in X^\omega \) is called strictly \( G \)-generic if for any \( v \), \( u \in X^\omega \) and any \( g \in G \) the transformation \( T_v g T_v^* \) either moves the point \( w \), or fixes \( w \) together with every point in a neighborhood of \( w \), or is not defined on \( w \).

Let \( w = \{x_i\}_i \in X^\omega \) be a strictly \( G \)-generic point. We claim that for any \( g \in G \) there exists \( n \in \mathbb{N} \) such that the product

\[ u_{x_n}^* \cdots u_{x_2}^* u_{x_1}^* gu_{x_2} u_{x_3} \cdots u_{x_n} = 0. \]

Assume to the contrary that for all \( n \) the above product is nonzero. Clearly this is the case if and only if

\[ g \cdot x_2 = x_1 \cdot g_2, \quad \text{for some } g_2 \in G, \text{ with} \]

\[ g_2 \cdot x_3 = x_2 \cdot g_3, \quad \text{for some } g_3 \in G, \text{ and so on} \]

\[ g_i \cdot x_{i+1} = x_i \cdot g_{i+1}, \quad \text{for all } i \geq 2, \text{ for elements } g_i \in G. \]

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This means that \( g : x_2 x_3 x_4 \ldots \mapsto x_1 x_2 x_3 x_4 \ldots \) The point \( w \) was strictly \( G \)-generic, and since the transformation \( gT_{x_1}^* \) fixes \( w \), we know there exists a neighborhood of \( w \) which is kept fixed by \( gT_{x_1}^* \), and in particular some cylindrical set \( x_1 \ldots x_m X^w \) which will be fixed. Taking any element \( x_1 \ldots x_m y_{m+1} y_{m+2} \ldots \) from this cylindrical set, we compute

\[
\begin{align*}
gT_{x_1}^* (x_1 \ldots x_m y_{m+1} y_{m+2} \ldots) &= g(x_2 \ldots x_m y_{m+1} y_{m+2} \ldots) \\
x_1 x_2 \ldots x_m g_{m+1} (y_{m+1} y_{m+2} \ldots) &= x_1 x_2 \ldots x_m y_{m+1} y_{m+2} \ldots
\end{align*}
\]

which, because of the faithfulness of the group action, implies that \( g_{m+1} = e \). Hence \( x_m = x_{m+i} \) for all \( i \geq 0 \) but this contradicts the fact that the point \( w \) was strictly \( G \)-generic, ergo there must exist some \( n \) such that the above product is zero.

**Proposition 6.7.** \( \Phi \) is nonperiodic.

**Proof.** Assume that \( \Phi \) is periodic. It suffices to consider periodicity for \( n = 1 \), i.e. \( \Phi \approx A_\Phi \) (the general case amounts to replacing the basis \( X \) by \( X^n \)). Then from the discussion above, extended from elements \( g \in G \) to general elements of \( CG \), we know that for any \( a \in CG \) there exists \( n \) such that

\[
u_{x_n}^* u_{x_{n-1}}^* \cdots u_{x_2}^* a u_{x_1} u_{x_2} \cdots u_{x_n} = 0.
\]

But on the other hand, for the element \( a = u_{x_1}^* \) we have

\[
u_{x_n}^* u_{x_{n-1}}^* u_{x_2}^* u_{x_1} u_{x_2} \cdots u_{x_n} = e,
\]

which is absurd. Hence \( \Phi \) cannot be periodic.

We are now able to give an alternative proof of Theorem 8.3 in [2].

**Theorem 6.8.** \( \mathcal{O}_\Phi \) is simple.

**Proof.** We have established in Proposition 6.5 and Proposition 6.7 that \( \Phi \) is minimal and nonperiodic, hence it follows from Theorem 5.7 that \( \mathcal{O}_\Phi \) is simple.
References


