1 Algebraic Geometry

Contents

1 Algebraic Geometry 1
  1.1 General terms .................................................. 2
  1.2 Moduli theory and stacks ...................................... 6
  1.3 Results and theorems .......................................... 7
  1.4 Sheaves and bundles ........................................... 9
  1.5 Singularities .................................................... 11
  1.6 Toric geometry .................................................. 12
  1.7 Types of varieties ............................................. 12

2 Commutative algebra 14
  2.1 Modules .......................................................... 14
  2.2 Results and theorems .......................................... 15
  2.3 Rings ............................................................. 16

3 Convex geometry 16
  3.1 Cones ............................................................. 16
  3.2 Polytopes ........................................................ 17

4 Homological algebra 18
  4.1 Classes of modules ............................................. 18
  4.2 Derived functors ................................................ 18

5 Differential and complex geometry 19
  5.1 Definitions and concepts ...................................... 19
  5.2 Results and theorems .......................................... 20
1.1 General terms

1.1.1 Cartier divisor

Let $\mathcal{K}_X$ be the sheaf of total quotients on $X$, and let $\mathcal{O}_X^*$ be the sheaf of non-zero divisors on $X$. We have an exact sequence

$$1 \to \mathcal{O}_X^* \to \mathcal{K}_X \to \mathcal{K}_X / \mathcal{O}_X^* \to 1.$$ 

Then a Cartier divisor is a global section of the quotient sheaf at the right.

1.1.2 Categorical quotient

Let $X$ be a scheme and $G$ a group. A categorical quotient is a morphism $\pi : X \to Y$ that satisfies the following two properties:

1. It is invariant, in the sense that $\pi \circ \sigma = \pi \circ p_2$ where $\sigma : G \times X \to X$ is the group action, and $p_2 : G \times X \to X$ is the projection. That is, the following diagram should commute:

$$\begin{array}{ccc}
G \times X & \xrightarrow{\sigma} & X \\
p_2 \downarrow & & \downarrow \pi \\
X & \xrightarrow{\pi} & Y \\
\end{array}$$

2. The map $\pi$ should be universal, in the following sense: If $\pi' : X \to Z$ is any morphism satisfying the previous condition, it should uniquely factor through $\pi$. That is:

$$\begin{array}{ccc}
X & \xrightarrow{\pi} & Y \\
\pi' & \downarrow h & \\
Z & \xrightarrow{\exists h} & \\
\end{array}$$

Note: A categorical quotient need not be surjective.
1.1.3 Chow group

Let $X$ be an algebraic variety. Let $Z_r(X)$ be the group of $r$-dimension cycles on $X$, a cycle being a $\mathbb{Z}$-linear combination of $r$-dimensional subvarieties of $X$. If $V \subset X$ is a subvariety of dimension $r + 1$ and $f : X \dashrightarrow \mathbb{A}^1$ is a rational function on $X$, then there is an integer $\text{ord}_W(f)$ for each codimension one subvariety of $V$, the order of vanishing of $f$. For a given $f$, there will only be finitely many subvarieties $W$ for which this number is non-zero. Thus we can define an element $[\text{div}(f)]$ in $Z_r(X)$ by $\sum \text{ord}_W(f)[W]$.

We say that two $r$-cycles $U_1, U_2$ are rationally equivalent if there exist $r + 1$-dimensional subvarieties $V_1, V_2$ together with rational functions $f_1 : V_1 \dashrightarrow \mathbb{A}^1, f_2 : V_2 \dashrightarrow \mathbb{A}^1$ such that $U_1 - U_2 = \sum [\text{div}(f_i)]$. The quotient group is called the Chow group of $r$-dimensional cycles on $X$, and denoted by $A_r(X)$.

1.1.4 Complete variety

Let $X$ be an integral, separated scheme over a field $k$. Then $X$ is complete if is proper.

Then $\mathbb{P}^n$ is proper over any field, and $\mathbb{A}^n$ is never proper.

1.1.5 Crepant resolution

A crepant resolution is a resolution of singularities $f : X \to Y$ that does not change the canonical bundle, i.e. such that $\omega_X \simeq f^*\omega_Y$.

1.1.6 Dominant map

A rational map $f : X \dashrightarrow Y$ is dominant if its image (or precisely: the image of one of its representatives) is dense in $Y$.

1.1.7 Étale map

A morphism of schemes of finite type $f : X \to Y$ is étale if it is smooth of dimension zero. This is equivalent to $f$ being flat and $\Omega_{X/Y} = 0$. This again is equivalent to $f$ being flat and unramified.

1.1.8 Genus

The geometric genus of a smooth, algebraic variety, is defined as the number of sections of the canonical sheaf, that is, as $H^0(V, \omega_X)$. This is often denoted $p_X$. 

3
1.1.9 Geometric quotient

Let $X$ be an algebraic variety and $G$ an algebraic group. Then a geometric quotient is a morphism of varieties $\pi : X \rightarrow Y$ such that

1. For each $y \in Y$, the fiber $\pi^{-1}(y)$ is an orbit of $G$.

2. The topology of $Y$ is the quotient topology: a subset $U$ of $Y$ is open if and only if $\pi^{-1}(U)$ is open.

3. For any open subset $U \subset Y$, $\pi^* : k[U] \rightarrow k[\pi^{-1}(U)]^G$ is an isomorphism of $k$-algebras.

The last condition may be rephrased as an isomorphism of structure sheaves: $\mathcal{O}_Y \simeq (\pi_* \mathcal{O}_X)^G$.

1.1.10 Hodge numbers

If $X$ is a complex manifold, then the Hodge numbers $h^{p,q}$ of $X$ are defined as the dimension of the cohomology groups $H^p(X, \Omega^q_X)$.

1.1.11 Linear series

A linear series on a smooth curve $C$ is the data $(\mathcal{L}, V)$ of a line bundle on $C$ and a vector subspace $V \subseteq H^0(C, \mathcal{L})$. We say that the linear series $(\mathcal{L}, V)$ have degree $\deg \mathcal{L}$ and rank $\dim V - 1$.

1.1.12 Log structure

A prelog structure on a scheme $X$ is given by a pair $(X, M)$, where $X$ is a scheme and $M$ is a sheaf of monoids on $X$ (on the Étale site) together with a morphisms $\alpha : M \rightarrow \mathcal{O}_X$. It is a log structure if the map $\alpha : \alpha^{-1} \mathcal{O}_X \rightarrow \mathcal{O}_X$ is an isomorphism.

See [5].

1.1.13 Néron-Severi group

Let $X$ be a nonsingular projective variety of dimension $\geq 2$. Then we can define the subgroup $\text{Cl}^0 X$ of $\text{Cl} X$, the subgroup consisting of divisor classes algebraically equivalent to zero. Then $\text{Cl} X / \text{Cl}^0 X$ is a finitely-generated group. It is denoted by $\text{NS}(X)$. 


1.1.14 Normal crossings divisor

Let $X$ be a smooth variety and $D \subset X$ a divisor. We say that $D$ is a **simple normal crossing divisor** if every irreducible component of $D$ is smooth and all intersections are transverse. That is, for every $p \in X$ we can choose local coordinates $x_1, \cdots, x_n$ and natural numbers $m_1, \cdots, m_n$ such that $D = (\prod_i x_i^{m_i} = 0)$ in a neighbourhood of $p$.

Then we say that a divisor is **normal crossing** (without the “simple”) if the neighbourhood above can is allowed to be chosen locally analytically or as a formal neighbourhood of $p$.

Example: the nodal curve $y^2 = x^3 + x^2$ is a a normal crossing divisor in $\mathbb{C}^2$, but not a simple normal crossing divisor.

This definition is taken from [6].

1.1.15 Normal variety

A variety $X$ is **normal** if all its local rings are normal rings.

1.1.16 Picard number

The **Picard number** of a nonsingular projective variety is the rank of Néron-Severi group.

1.1.17 Proper morphism

A morphism $f : X \to Y$ is **proper** if it separated, of finite type, and universally closed.

1.1.18 Resolution of singularities

A morphism $f : X \to Y$ is a **resolution of singularities** of $Y$ if $X$ is non-singular and $f$ is birational and proper.

1.1.19 Separated

Let $f : X \to Y$ be a morphism of schemes. Let $\Delta : X \to X \times_Y X$ be the diagonal morphism. We say that $f$ is **separated** if $\Delta$ is a closed immersion. We say that $X$ is **separated** if the unique morphism $f : X \to \text{Spec} \mathbb{Z}$ is separated.

This is equivalent to the following: for all open affines $U, V \subset X$, the intersection $U \cap V$ is affine and $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ generate $\mathcal{O}_X(U \cap V)$. For example: let $X = \mathbb{P}^1$ and let $U_1 = \{[x : 1]\}$ and $U_2 = \{[1 : y]\}$. Then
\( \mathcal{O}_X(U_1) = \text{Spec } k[x] \) and \( \mathcal{O}_X(U_2) = \text{Spec } k[y] \). The glueing map is given on the ring level as \( x \mapsto \frac{1}{y} \). Then \( \mathcal{O}_X(U_1 \cap U_2) = k[y, \frac{1}{y}] \).

### 1.1.20 Unirational variety

A variety \( X \) is unirational if there exists a generically finite dominant map \( \mathbb{P}^n \rightarrow X \).

### 1.2 Moduli theory and stacks

#### 1.2.1 Étale site

Let \( S \) be a scheme. Then the small étale site over \( S \) is the site, denoted by \( \text{Ét}(S) \) that consists of all étale morphisms \( U \rightarrow S \) (morphisms being commutative triangles). Let \( \text{Cov}(U \rightarrow S) \) consist of all collections \( \{U_i \rightarrow U\}_{i \in I} \) such that

\[
\prod_{i \in I} U_i \rightarrow U
\]

is surjective.

#### 1.2.2 Grothendieck topology

Let \( C \) be a category. A Grothendieck topology on \( C \) consists of a set \( \text{Cov}(X) \) of sets of morphisms \( \{X_i \rightarrow X\}_{i \in I} \) for each \( X \) in \( \text{Ob}(C) \), satisfying the following axioms:

1. If \( V \xrightarrow{\cong} X \) is an isomorphism, then \( \{V \rightarrow X\} \in \text{Cov}(X) \).
2. If \( \{X_i \rightarrow X\}_{i \in I} \in \text{Cov}(X) \) and \( Y \rightarrow X \) is a morphism in \( C \), then the fiber products \( X_i \times_X Y \) exists and \( \{X_i \times_X Y \rightarrow Y\}_{i \in I} \in \text{Cov}(Y) \).
3. If \( \{X_i \in X\}_{i \in I} \in \text{Cov}(X) \), and for each \( i \in I \), \( \{V_{ij} \rightarrow X_i\}_{j \in J} \in \text{Cov}(X_i) \), then

\[
\{V_{ij} \rightarrow X_i \rightarrow X\}_{i \in I, j \in J} \in \text{Cov}(X).
\]

The easiest example is this: Let \( C \) be the category of open sets on a topological space \( X \), the morphisms being only the inclusions. Then for each \( U \in \text{Ob}(C) \), define \( \text{Cov}(U) \) to be the set of all coverings \( \{U_i \rightarrow U\}_{i \in I} \) such that \( U = \bigcup_{i \in I} U_i \). Then it is easily checked that this defines a Grothendieck topology.
1.2.3 Site

A site is a category equipped with a Grothendieck topology.

1.3 Results and theorems

1.3.1 Adjunction formula

Let $X$ be a smooth algebraic variety $Y$ a smooth subvariety. Let $i : Y \hookrightarrow X$ be the inclusion map, and let $\mathcal{I}$ be the corresponding ideal sheaf. Then $
abla_Y = i^* \nabla_X \otimes_{\mathcal{O}_X} \det(\mathcal{I}/\mathcal{I}^2)^\vee$, where $\nabla_Y$ is the canonical sheaf of $Y$.

In terms of canonical classes, the formula says that $K_D = (K_X + D)|_D$.

Here’s an example: Let $X$ be a smooth quartic surface in $\mathbb{P}^3$. Then $H^1(X, \mathcal{O}_X) = 0$. The divisor class group of $\mathbb{P}^3$ is generated by the class of a hyperplane, and $K_{\mathbb{P}^3} = -4H$. The class of $X$ is then $4H$ since $X$ is of degree 4. $X$ corresponds to a smooth divisor $D$, so by the adjunction formula, we have that

$$K_D = (K_{\mathbb{P}^3} + D)|_D = -4H + 4H|_D = 0.$$

Thus $X$ is an example of a K3 surface.

1.3.2 Bertini’s Theorem

Let $X$ be a nonsingular closed subvariety of $\mathbb{P}^n_k$, where $k = \bar{k}$. Then the set of of hyperplanes $H \subseteq \mathbb{P}^n_k$ such that $H \cap X$ is regular at every point) and such that $H \not\subseteq X$ is a dense open subset of the complete linear system $|H|$. See [4, Thm II.8.18].

1.3.3 Chow’s lemma

Chow’s lemma says that if $X$ is a scheme that is proper over $k$, then it is “fairly close” to being projective. Specifically, we have that there exists a projective $k$-scheme $X'$ and morphism $f : X' \to X$ that is birational.

So every scheme proper over $k$ is birational to a projective scheme. For a proof, see for example the Wikipedia page.

1.3.4 Euler sequence

If $A$ is a ring and $\mathbb{P}^n_A$ is projective $n$-space over $A$, then there is an exact sequence of sheaves on $X$:

$$0 \to \Omega_{\mathbb{P}^n_A/A} \to \mathcal{O}_{\mathbb{P}^n_A}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n_A} \to 0.$$
1.3.5 Genus-degree formula

If $C$ is a smooth plane curve, then its genus can be computed as

$$g_C = \frac{(d-1)(d-2)}{2}.$$  

This follows from the adjunction formula. In particular, there are no curves of genus 2 in the plane.

1.3.6 Hirzebruch-Riemann-Roch formula

Let $X$ be a nonsingular variety and let $\mathcal{T}_X$ be its tangent bundle. Let $\mathcal{E}$ be a locally free sheaf on $X$. Then

$$\chi(\mathcal{E}) = \deg (\text{ch}(\mathcal{E}) \cdot \text{td}(\mathcal{T}))_n,$$

where $\chi$ is the Euler characteristic, ch denotes the Chern class, and td denotes the Todd class. See [4, Appendix A].

1.3.7 Hurwitz' formula

Let $X, Y$ be smooth curves in the sense of Hartshorne. That is, they are integral 1-dimensional schemes, proper over a field $k$ (with $\overline{k} = k$), all of whose local rings are regular.

Then Hurwitz’ formula says that if $f : X \to Y$ is a separable morphism and $n = \deg f$, then

$$2(g_X - 1) = 2n(g_Y - 1) + \deg R,$$

where $R$ is the ramification divisor of $f$, and $g_X, g_Y$ are the genera of $X$ and $Y$, respectively. See Example 6.1.1.

1.3.8 Kodaira vanishing

If $k$ is a field of characteristic zero, $X$ is a smooth and projective $k$-scheme of dimension $d$, and $\mathcal{L}$ is an ample invertible sheaf on $X$, then $H^q(X, \mathcal{L} \otimes \mathcal{O}_X \Omega^p_{X/k}) = 0$ for $p + q > d$. In addition, $H^q(X, \mathcal{L}^{-1} \otimes \mathcal{O}_X \Omega^p_{X/k}) = 0$ for $p + q < d$.

1.3.9 Lefschetz hyperplane theorem

Let $X$ be an $n$-dimensional complex projective algebraic variety in $\mathbb{P}^d$ and let $Y$ be a hyperplane section of $X$ such that $U = X \setminus Y$ is smooth. Then the natural map $H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$ in singular cohomology is an isomorphism for $k < n - 1$ and injective for $k = n - 1$. 

8
1.3.10 Riemann-Roch for curves

The Riemann-Roch theorem relates the number of sections of a line bundle with the genus of a smooth proper curve $C$. Let $\mathcal{L}$ be a line bundle $\omega_C$ the canonical sheaf on $C$. Then

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^{-1} \otimes_{\mathcal{O}_C} \omega_C) = \deg(\mathcal{L}) + 1 - g.$$ 

This is [4, Theorem IV.1.3].

1.3.11 Semi-continuity theorem

Let $f : X \to Y$ be a projective morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on $X$, flat over $Y$. Then for each $i \geq 0$, the function $h^i(y, \mathcal{F}) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$ is an upper semicontinuous function on $Y$. See [4, Chapter III, Theorem 12.8].

1.3.12 Serre duality

Let $X$ be a projective Cohen-Macaulay scheme of equidimension $n$. Then for any locally free sheaf $\mathcal{F}$ on $X$ there are natural isomorphisms

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\vee).$$

Here $\omega_X^\vee$ is a dualizing sheaf for $X$. In the case that $X$ is nonsingular, we have that $\omega_X^\vee \cong \omega_X$, the canonical sheaf on $X$ (see [4, Chapter III, Corollary 7.12]).

1.3.13 Serre vanishing

One form of Serre vanishing states that if $X$ is a proper scheme over a noetherian ring $A$, and $\mathcal{L}$ is an ample sheaf, then for any coherent sheaf $\mathcal{F}$ on $X$, there exists an integer $n_0$ such that for each $i > 0$ and $n \geq n_0$ the group $H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = 0$ vanishes. See [4, Proposition III.5.3].

1.4 Sheaves and bundles

1.4.1 Ample line bundle

A line bundle $\mathcal{L}$ is ample if for any coherent sheaf $\mathcal{F}$ on $X$, there is an integer $n$ (depending on $\mathcal{F}$) such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^\otimes n$ is generated by global sections. Equivalently, a line bundle $\mathcal{L}$ is ample if some tensor power of it is very ample.
1.4.2 Invertible sheaf
A locally free sheaf of rank 1 is called invertible. If \( X \) is normal, then, invertible sheaves are in 1–1 correspondence with line bundles.

1.4.3 Anticanonical sheaf
The anticanonical sheaf \( \omega_{-1}^X \) is the inverse of the canonical sheaf \( \omega_X \), that is \( \omega_{-1}^X = \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X) \).

1.4.4 Canonical class
The canonical class \( K_X \) is the class of the canonical sheaf \( \omega_X \) in the divisor class group.

1.4.5 Canonical sheaf
If \( X \) is a smooth algebraic variety of dimension \( n \), then the canonical sheaf is \( \omega := \wedge^n \Omega^1_{X/k} \) the \( n \)'th exterior power of the cotangent bundle of \( X \).

1.4.6 Nef divisor
Let \( X \) be a normal variety. Then a Cartier divisor \( D \) on \( X \) is nef (numerically effective) if \( D \cdot C \geq 0 \) for every irreducible complete curve \( C \subseteq X \). Here \( D \cdot C \) is the intersection product on \( X \) defined by \( \text{deg}(\phi^* \mathcal{O}_X(D)) \). Here \( \phi : C' \rightarrow C \) is the normalization of \( C \).

1.4.7 Sheaf of holomorphic \( p \)-forms
If \( X \) is a complex manifold, then the sheaf of of holomorphic \( p \)-forms \( \Omega^p_X \) is the \( p \)-th wedge power of the cotangent sheaf \( \wedge^p \Omega^1_X \).

1.4.8 Normal sheaf
Let \( Y \hookrightarrow X \) be a closed immersion of schemes, and let \( \mathcal{I} \subseteq \mathcal{O}_X \) be the ideal sheaf of \( Y \) in \( X \). Then \( \mathcal{I}/\mathcal{I}^2 \) is a sheaf on \( Y \), and we define the sheaf \( \mathcal{N}_{Y/X} \) by \( \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \).

1.4.9 Rank of a coherent sheaf
Given a coherent sheaf \( \mathcal{F} \) on an irreducible variety \( X \), form the sheaf \( \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X \). Its global sections is a finite dimensional vector space, and we say that \( \mathcal{F} \) has rank \( r \) if \( \dim_k \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X) = r \).
1.4.10 Reflexive sheaf

A sheaf $\mathcal{F}$ is reflexive if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism. Here $\mathcal{F}^{\vee}$ denotes the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

1.4.11 Very ample line bundle

A line bundle $\mathcal{L}$ is very ample if there is an embedding $i : X \hookrightarrow \mathbb{P}^n_S$ such that the pullback of $\mathcal{O}_{\mathbb{P}^n_S}(1)$ is isomorphic to $\mathcal{L}$. In other words, there should be an isomorphism $i^* \mathcal{O}_{\mathbb{P}^n_S}(1) \simeq \mathcal{L}$.

1.5 Singularities

1.5.1 Canonical singularities

A variety $X$ has canonical singularities if it satisfies the following two conditions:

1. For some integer $r \geq 1$, the Weil divisor $rK_X$ is Cartier (equivalently, it is $\mathbb{Q}$-Cartier).

2. If $f : Y \to X$ is a resolution of $X$ and $\{E_i\}$ the exceptional divisors, then

$$rK_Y = f^*(rK_X) + \sum a_iE_i$$

with $a_i \geq 0$.

The integer $r$ is called the index, and the $a_i$ are called the discrepancies at $E_i$.

1.5.2 Terminal singularities

A variety $X$ have terminal singularities if the $a_i$ in the definition of canonical singularities are all greater than zero.

1.5.3 Ordinary double point

An ordinary double point is a singularity that is analytically isomorphic to $x^2 = yz$. 

11
1.6 Toric geometry

1.6.1 Chow group of a toric variety

The Chow group $A_{n-1}(X)$ of a toric variety can be computed directly from its fan. Let $\Sigma(1)$ be the set of rays in $\Sigma$, the fan of $X$. Then we have an exact sequence

$$0 \rightarrow M \rightarrow \mathbb{Z}^{\Sigma(1)} \rightarrow A_{n-1}(X) \rightarrow 0.$$ 

The first map is given by sending $m \in M$ to $(\langle m, v_\rho \rangle)_{\rho \in \Sigma(1)}$, where $v_\rho$ is the unique generator of the semigroup $\rho \cap N$. The second map is given by sending $(a_\rho)_{\rho \in \Sigma(1)}$ to the divisor class of $\sum_\rho a_\rho D_\rho$.

1.6.2 Generalized Euler sequence

The generalized Euler sequence is a generalization of the Euler sequence for toric varieties. If $X$ is a smooth toric variety, then its cotangent bundle $\Omega^1_X$ fits into an exact sequence

$$0 \rightarrow \Omega^1_X \rightarrow \bigoplus_\rho \mathcal{O}_X(-D_\rho) \rightarrow \text{Pic}(X) \otimes \mathbb{Z} \mathcal{O}_X \rightarrow 0.$$ 

Here $D_\rho$ is the divisor corresponding to the ray $\rho \in \Sigma(1)$. See [2, Chapter 8].

1.6.3 Polarized toric variety

A toric variety equipped with an ample $T$-invariant divisor.

1.6.4 Toric variety associated to a polytope

There are several ways to do this. Here is one: Let $\Delta \subset M_\mathbb{R}$ be a convex polytope. Embed $\Delta$ in $M_\mathbb{R} \times \mathbb{R}$ by $\Delta \times \{1\}$ and let $C_\Delta$ be the cone over $\Delta \times \{1\}$, and let $\mathbb{C}[C_\Delta \cap (M \times \mathbb{Z})]$ be the corresponding semigroup ring. This is a semigroup ring graded by the $\mathbb{Z}$-factor. Then we define $\mathbb{P}_\Delta = \text{Proj} \mathbb{C}[C_\Delta \cap (M \times \mathbb{Z})]$ to be the toric variety associated to a polytope.

1.7 Types of varieties

1.7.1 Abelian variety

A variety $X$ is an abelian variety if it is a connected and complete algebraic group over a field $k$. Examples include elliptic curves and for special lattices $\Lambda \subset \mathbb{C}^{2g}$, the quotient $\mathbb{C}^{2g}/\Lambda$ is an abelian variety.
1.7.2 Calabi-Yau variety

In algebraic geometry, a Calabi-Yau variety is a smooth, proper variety $X$ over a field $k$ such that the canonical sheaf is trivial, that is, $\omega_X \cong \mathcal{O}_X$, and such that $H^j(X, \mathcal{O}_X) = 0$ for $1 \leq j \leq n - 1$.

1.7.3 del Pezzo surface

A del Pezzo surface is a 2-dimensional Fano variety. In other words, they are complete non-singular surfaces with ample anticanonical bundle. The degree of the del Pezzo surface $X$ is by definition the self intersection number $K.K$ of its canonical class $K$.

1.7.4 Elliptic curve

An elliptic curve is a smooth, projective curve of genus 1. They can all be obtained from an equation of the form $y^2 = x^3 + ax + b$ such that $\Delta = -2^4(4a^3 + 27b^2) \neq 0$.

1.7.5 Elliptic surface

An elliptic surface is a smooth surface $X$ with a morphism $\pi : X \to B$ onto a non-singular curve $B$ whose generic fiber is a non-singular elliptic curve.

1.7.6 Fano variety

A variety $X$ is Fano if the anticanonical sheaf $\omega_X^{-1}$ is ample.

1.7.7 Jacobian variety

Let $X$ be a curve of genus $g$ over $k$. The Jacobian variety of $X$ is a scheme $J$ of finite type over $k$, together with an element $\mathcal{L} \in \text{Pic}^0(X/J)$, with the following universal property: for any scheme $T$ of finite type over $k$ and for any $\mathcal{M} \in \text{Pic}^0(X/T)$, there is a unique morphism $f : T \to J$ such that $f^*\mathcal{L} \cong \mathcal{M}$ in $\text{Pic}^0(X/T)$. This just says that $J$ represents the functor $T \mapsto \text{Pic}^0(X/T)$.

If $J$ exists, its closed points are in $1-1$ correspondence with elements of $\text{Pic}^0(X)$. It can be checked that $J$ is actually a group scheme. For details, see [4, Ch. IV.4].
1.7.8 K3 surface

A K3 surface is a complex algebraic surface $X$ such that the canonical sheaf is trivial, $\omega_X \cong \mathcal{O}_X$, and such that $H^1(X, \mathcal{O}_X) = 0$. These conditions completely determine the Hodge numbers of $X$.

1.7.9 Stanley-Reisner scheme

A Stanley-Reisner scheme is a projective variety associated to a simplicial complex as follows. Let $K$ be a simplicial complex. Then we define an ideal $I_K \subseteq k[x_v | v \in V(K)] = k[x]$ (here $V(K)$ denotes the vertex set of $K$) by

$$I_K = \langle x_{v_{i_1}} x_{v_{i_2}} \cdots x_{v_{i_k}} | v_{i_1} v_{i_2} \cdots v_{i_k} \notin K \rangle.$$  

We get a projective scheme $\mathbb{P}(K)$ defined by $\text{Proj} \left( k[x]/I_K \right)$, together with an embedding into $\mathbb{P}^{\#V(K)-1}$. It can be shown that $H^p(\mathbb{P}(K), \mathcal{O}_{\mathbb{P}(K)}) \cong H^p(K; k)$, where the right-hand-side denotes the cohomology group of the simplicial complex.

1.7.10 Toric variety

A toric variety $X$ is an integral scheme containing the torus $(k^*)^n$ as a dense open subset, such that the action of the torus on itself extends to an action $(k^*)^n \times X \to X$.

2 Commutative algebra

2.1 Modules

2.1.1 Depth

Let $R$ be a noetherian ring, and $M$ a finitely-generated $R$-module and $I$ an ideal of $R$ such that $IM \neq M$. Then the $I$-depth of $M$ is (see $\text{Ext}$):

$$\inf \{ i \mid \text{Ext}_R^i(R/I, M) \neq 0 \}.$$  

This is also the length of a maximal $M$-sequence in $I$.

2.1.2 Rank

If $R$ has the invariant basis property (IBN), then we define the rank of a free module to be the cardinality of any basis.
2.1.3 Kähler differentials

Let $A \to B$ be a ring homomorphism. The **module of Kähler differentials** $\Omega_{B/A}$ is the module together with a map $d : B \to \Omega_{B/A}$ satisfying the following universal property: if $D : B \to M$ is any $A$-linear derivation (an element of $\text{Der}_A(B, M)$), then there is a unique module homomorphism $\bar{D} : \Omega_{B/A} \to M$ such that

\[
\begin{array}{ccc}
B & \xrightarrow{d} & \Omega_{B/A} \\
\downarrow{D} & & \downarrow{\bar{D}} \\
M & & 
\end{array}
\]

is commutative. Thus we have a natural isomorphism $\text{Der}_A(B, M) = \text{Hom}_B(\Omega_{B/A}, M)$. In the language of category theory, this means that $\text{Der}_A(B, -)$ is corepresented by $\Omega_{B/A}$.

A concrete construction of $\Omega_{B/A}$ is given as follows. Let $M$ be the free $B$-module generated by all symbols $df$, where $f \in B$. Let $N$ be the submodule generated by $da$ if $a \in A$, $d(f + g) - df - dg$ and the Leibniz rule $d(fg) = fdg + gdf$. Then $M/N \simeq \Omega_{B/A}$ as $B$-modules.

2.2 Results and theorems

2.2.1 The conormal sequence

The **conormal sequence** is a sequence relating Kähler differentials in different rings. Specifically, if $A \to B \to 0$ is a surjection of rings with kernel $I$, then we have an exact sequence of $B$-modules:

\[
\begin{array}{c}
I/I^2 \xrightarrow{d} B \otimes_A \Omega_{B/A} \xrightarrow{D\pi} \Omega_{T/R} \to 0
\end{array}
\]

The map $d$ sends $f \mapsto 1 \otimes df$, and $D\pi$ sends $c \otimes db \mapsto cdb$. For proof, see [3, Chapter 16].

2.2.2 The Unmixedness Theorem

Let $R$ be a ring. If $I = \langle x_1, \cdots, x_n \rangle$ is an ideal generated by $n$ elements such that $\text{codim} I = n$, then all minimal primes of $I$ have codimension $n$. If in addition $R$ is Cohen-Macaulay, then every associated prime of $I$ is minimal over $I$. See the discussion after [3, Corollary 18.14] for more details.
2.3 Rings

2.3.1 Cohen-Macaulay ring

A local Cohen-Macaulay ring (CM-ring for short) is a commutative noetherian local ring with Krull dimension equal to its depth. A ring is Cohen-Macaulay if its localization at all prime ideals are Cohen-Macaulay.

2.3.2 Depth of a ring

The depth of a ring $R$ is is its depth as a module over itself.

2.3.3 Gorenstein ring

A commutative ring $R$ is Gorenstein if each localization at a prime ideal is a Gorenstein local ring. A Gorenstein local ring is a local ring with finite injective dimension as an $R$-module. This is equivalent to the following: $\text{Ext}^i_R(k, R) = 0$ for $i \neq n$ and $\text{Ext}^n_R(k, R) \simeq k$ (here $k = R/m$ and $n$ is the Krull dimension of $R$).

2.3.4 Invariant basis property

A ring $R$ satisfies the invariant basis property (IBP) if $R^n \not\cong R^{n+t}$ $R$-modules for any $t \neq 0$.

2.3.5 Normal ring

An integral domain $R$ is normal if all its localizations at prime ideals $p \in \text{Spec } R$ are integrally closed domains.

3 Convex geometry

3.1 Cones

3.1.1 Gorenstein cone

A strongly convex cone $C \subset M_{\mathbb{R}}$ is Gorenstein if there exists a point $n \in N$ in the dual lattice such that $\langle v, n \rangle = 1$ for all generators of the semigroup $C \cap M$. 

16
3.1.2 Reflexive Gorenstein cone

A cone $C$ is reflexive if both $C$ and its dual $C^\vee$ are Gorenstein cones. See for example [1].

3.1.3 Simplicial cone

A cone $C$ generated by $\{v_1, \cdots, v_k\} \subseteq \mathbb{N}_\mathbb{R}$ is simplicial if the $v_i$ are linearly independent.

3.2 Polytopes

3.2.1 Dual (polar) polytope

If $\Delta$ is a polyhedron, its dual $\Delta^\circ$ is defined by

$$\Delta^\circ = \{x \in \mathbb{N}_\mathbb{R} \mid \langle x, y \rangle \geq -1 \forall y \in \Delta\}.$$ 

3.2.2 Gorenstein polytope of index $r$

A lattice polytope $P \subset \mathbb{R}^{d+r-1}$ is called a Gorenstein polytope of index $r$ if $rP$ contains a single interior lattice point $p$ and $rP - p$ is a reflexive polytope.

3.2.3 Nef partition

Let $\Delta \subset M_\mathbb{R}$ be a $d$-dimensional reflexive polytope, and let $m = \text{int}(\Delta) \cap M$. A Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ where $\Delta_1, \ldots, \Delta_r$ are lattice polytopes is called a nef partition of $\Delta$ of length $r$ if there are lattice points $p_i \in \Delta_i$ for all $i$ such that $p_1 + \cdots + p_r = m$. The nef partition is called centered if $p_i = 0$ for all $i$.

This is equivalent to the toric divisor $D_j = \mathcal{O}(\Delta_i) = \sum_{\rho \in \Delta_i} D_\rho$ being a Cartier divisor generated by its global sections. See [1, Chapter 4.3].

3.2.4 Reflexive polytope

A polytope $\Delta$ is reflexive if the following two conditions hold:

1. All facets $\Gamma$ of $\Delta$ are supported by affine hyperplanes of the form $\{m \in M_\mathbb{R} \mid \langle m, v_\Gamma \rangle = -1\}$ for some $v_\Gamma \in N$.

2. The only interior point of $\Delta$ is 0, that is: $\text{Int}(\Delta) \cap M = \{0\}$.

It can be proved that a polytope $\Delta$ is reflexive if and only if the associated toric variety $\mathbb{P}_\Delta$ is Fano.
4 Homological algebra

4.1 Classes of modules

4.1.1 Projective modules

Projective modules are those satisfying a universal lifting property. A module \( P \) is **projective** if for every epimorphism \( \alpha : M \to N \) and every map, \( \beta : P \to N \), there exists a map \( \gamma : P \to M \) such that \( \beta = \alpha \circ \gamma \).

\[
\begin{array}{c}
\exists \gamma \\
\downarrow \\
\beta \\
M \xrightarrow{\alpha} N \xrightarrow{\beta} 0
\end{array}
\]

These are the modules \( P \) such that \( \text{Hom}(P, -) \) is exact.

4.2 Derived functors

4.2.1 Ext

Let \( R \) be a ring and \( M, N \) be \( R \)-modules. Then \( \text{Ext}^i_R(M, N) \) is the right-derived functors of the \( \text{Hom}(M, -) \)-functor. In particular, \( \text{Ext}^i_R(M, N) \) can be computed as follows: choose a projective resolution \( C \) of \( N \) over \( R \). Then apply the left-exact functor \( \text{Hom}_R(M, -) \) to the resolution and take homology. Then \( \text{Ext}^i_R(M, N) = h^i(C) \).

4.2.2 Local cohomology

Let \( R \) be a ring and \( I \subset R \) an ideal. Let \( \Gamma_I(-) \) be the following functor on \( R \)-modules:

\[
\Gamma_I(M) = \{ f \in M \mid \exists n \in \mathbb{N}, \text{s.t. } \Gamma^n f = 0 \}.
\]

Then \( H^i_I(-) \) is by definition the \( i \)th right derived functor of \( \Gamma_I \). In the case that \( R \) is noetherian, we have \( H^i_I(M) = \lim_{\longrightarrow} \text{Ext}^i_R(R/I^n, M) \).


4.2.3 Tor

Let \( R \) be a ring and \( M, N \) be \( R \)-modules. Then \( \text{Tor}^i_R(M, N) \) is the right-derived functors of the \( - \otimes_R N \)-functor. In particular \( \text{Tor}^i_R(M, N) \) can be computed by taking a projective resolution of \( M \), tensoring with \( N \), and then taking homology.
5 Differential and complex geometry

5.1 Definitions and concepts

5.1.1 Almost complex structure

An almost complex structure on a manifold $M$ is a map $J : T(M) \to T(M)$ whose square is $-1$. 

5.1.2 Connection

Let $E \to M$ be a vector bundle over $M$. A connection is a $\mathbb{R}$-linear map $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ such that the Leibniz rule holds:

$$\nabla(f\sigma) = f\nabla(\sigma) + \sigma \otimes df$$

for all functions $f : M \to \mathbb{R}$ and sections $\sigma \in \Gamma(E)$. 

5.1.3 Hermitian manifold

A Hermitian metric on a complex vector bundle $E$ over a manifold $M$ is a positive-definite Hermitian form on each fiber. Such a metric can be written as a smooth section $\Gamma(E \otimes \bar{E})^*$, such that $h_p(\eta, \bar{\zeta}) = h_p(\bar{\zeta}, \bar{\eta})$ for all $p \in M$, and such that $h_p(\eta, \bar{\eta}) > 0$ for all $p \in M$. A Hermitian manifold is a complex manifold with a Hermitian metric on its holomorphic tangent space $T^{(1,0)}(M)$. 

5.1.4 Kähler manifold

A Kahler manifold is ????

5.1.5 Symplectic manifold

A 2n-dimensional manifold $M$ is symplectic if it is compact and oriented and has a closed real two-form $\omega \in \bigwedge^2 T^*(M)$ which is nondegenerate, in the sense that $\bigwedge^n \omega\big|_p \neq 0$ for all $p \in M$. 

5.2 Results and theorems

6 Worked examples

6.1 Algebraic geometry

6.1.1 Hurwitz formula and Kähler differentials

Let $X$ be the conic in $\mathbb{P}^2$ given with ideal sheaf $\langle xz - y^2 \rangle$. Let $Y$ be $\mathbb{P}^1$, and consider the map $f : X \to Y$ given by projection onto the $xz$-line. $X$ is covered by two affine pieces, namely $X = U_x \cup U_z$, the spectra of the homogeneous localizations at $x, z$, respectively. Let $U_x = \text{Spec } A$ for $A = k[z]$ and $U_z = \text{Spec } B$ for $B = k[x]$. Then the map is locally given by $A \to k[y, z]/(z - y^2)$ where $z \mapsto \bar{z}$, and similarly for $B$. We have an isomorphism $k[y, z]/(z - y^2) \cong k[t]$, given by $y \mapsto t$ and $z \mapsto t^2$, so that locally the map is given by $k[z] \to k[t], z \mapsto t^2$.

This is a map of smooth projective curves, so we can apply Hurwitz’ formula. Both $X, Y$ are $\mathbb{P}^1$, so both have genus zero. Hence Hurwitz formula says that

$$-2 = -n \cdot 2 + \deg R,$$

where $R$ is the ramification divisor and $n$ is the degree of the map. The degree of the map can be defined locally, and it is the degree of the field extension $k(Y) \hookrightarrow k(X)$. But (the image of) $k(Y) = k(t^2)$ and $k(X) = k(t)$, so that $[k(Y) : k(X)] = 2$. Hence by Hurwitz’ formula, we should have $\deg R = 2$. Since $R = \sum_{P \in X} \text{length } \Omega_{X/Y, P}$, we should look at the sheaf of relative differentials $\Omega_{X/Y}$.

First we look in the chart $U_z$. We compute that $\Omega_{k[t]/k[z]} = k[t]/(t)$. This follows from the relation $d(t^2) = 2dt$, implying that $dt = 0$ in $\Omega_{k[t]/k[z]}$. This module is zero localized at all primes but $(t)$, where it is $k$. Thus for $P = (0 : 1 : 0)$, we have length $\Omega_{X/Y, P} = 1$.

The situation is symmetric with $z \leftrightarrow x$, so that we have $R = (0 : 0 : 1) + (1 : 0 : 0)$, confirming that $\deg R = 2$.

In fact, the curve $C$ is isomorphic to $\mathbb{P}^1$ via the map $\mathbb{P}^1 \to C$ given by $(s : t) \mapsto (s^2 : st : t^2)$. Identifying $C$ with $\mathbb{P}^1$, we thus see that $C \to \mathbb{P}^1$ correspond to the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $(s : t) \mapsto (s^2 : t^2)$.

6.2 The quintic threefold

Let $Y$ be a the zeroes of a general hypersurface of degree 5 in $\mathbb{P}^4$, or in other words, a section of $\omega_{\mathbb{P}^4}$. We want to compute the cohomology of $Y$ and its Hodge numbers. Let $\mathbb{P} = \mathbb{P}^4$. 
We have the ideal sheaf sequence

\[ 0 \to \mathcal{I} \to \mathcal{O}_\mathbb{P} \to i^* \mathcal{O}_Y \to 0, \]

where \( i : Y \to \mathbb{P}^4 \) is the inclusion. Note that \( \mathcal{I} = \mathcal{O}_\mathbb{P}(-5) \). Thus we have from the long exact sequence of cohomology that

\[ \cdots \to H^i(\mathbb{P}, \mathcal{I}) \to H^i(\mathbb{P}, \mathcal{O}_\mathbb{P}) \to H^i(Y, \mathcal{O}_Y) \to H^{i+1}(\mathbb{P}, \mathcal{I}) \to \cdots \]

Note that \( H^{i+1}(\mathbb{P}, \mathcal{I}) = 0 \) for \( i \neq 3 \) and 1 for \( i = 3 \). Also \( H^i(\mathbb{P}, \mathcal{O}_\mathbb{P}) = 0 \) unless \( i = 0 \) in which case it is 1. Thus we get that \( H^i(Y, \mathcal{O}_Y) \) is \( k \) for \( i = 0 \), for \( i = 1, 2 \) it is 0, and for \( i = 3 \) it is \( k \). For higher \( i \) it is zero by Grothendieck vanishing.

The adjunction formula relates the canonical bundles as follows: if \( \omega_{\mathbb{P}} \) is the canonical bundle on \( \mathbb{P} \), then \( \omega_Y = i^* \omega_{\mathbb{P}} \otimes \mathcal{O}_Y \det(\mathcal{I}/\mathcal{I}^2)^\vee \). The ideal sheaf is already a line bundle, so taking the determinant does not change anything. Now \( (\mathcal{I}/\mathcal{I}^2)^\vee = \operatorname{Hom}_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) = \operatorname{Hom}_{\mathbb{P}}(\mathcal{I}, \mathcal{O}_Y) = \operatorname{Hom}_{\mathbb{P}}(\mathcal{O}_\mathbb{P}(-5), \mathcal{O}_Y) = \mathcal{O}_Y(5) \).

It follows that \( \omega_Y = \mathcal{O}_Y(-5) \otimes \mathcal{O}_Y(5) = \mathcal{O}_Y \). Thus the canonical bundle is trivial and we conclude that \( Y \) is Calabi-Yau.

It remains to compute the Hodge numbers. We start with \( h^{11} = \dim_k H^1(Y, \Omega_Y) \).

We have the conormal sequence of sheaves on \( Y \):

\[ 0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y \to \Omega_Y \to 0, \]

which gives us the long exact sequence:

\[ \cdots \to H^i(\mathcal{I}/\mathcal{I}^2) \to H^i(\Omega_{\mathbb{P}} \otimes \mathcal{O}_Y) \to H^i(\Omega_Y) \to H^{i+1}(\mathcal{I}/\mathcal{I}^2) \to \cdots \]

We first compute the cohomology of \( \mathcal{I}/\mathcal{I}^2 \). We use the short exact sequence

\[ 0 \to \mathcal{O}_\mathbb{P}(-10) \to \mathcal{O}_\mathbb{P}(-5) \to \mathcal{I}/\mathcal{I}^2 \to 0. \] (1)

we have \( H^i(\mathbb{P}, \mathcal{O}_\mathbb{P}(-10)) = 0 \) for \( i = 0, 1, 2, 3 \), and for \( i = 4 \) we have \( H^4(\mathbb{P}, \mathcal{O}_\mathbb{P}(-10)) = H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(5)) = k^{125} \). Similarly \( H^i(\mathbb{P}, \mathcal{O}_\mathbb{P}(-5)) = 0 \) for \( i = 0, 1, 2, 3 \) and \( H^1(\mathbb{P}, \mathcal{O}_\mathbb{P}(-5)) = H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}) = k \). We conclude that \( h^i(Y, \mathcal{I}/\mathcal{I}^2) = 0 \) for \( i = 0, 1, 2 \) and 125 for \( i = 3 \).

In particular \( H^1(\Omega_Y) \simeq H^1(\Omega_{\mathbb{P}} \otimes \mathcal{O}_Y) \). We have the Euler sequence:

\[ 0 \to \Omega_{\mathbb{P}} \to \mathcal{O}_\mathbb{P}(-1)^{\oplus 5} \to \mathcal{O}_\mathbb{P} \to 0 \]
Now $\mathcal{O}_Y = \mathcal{O}_\mathbb{P} / \mathcal{I}$ is a flat $\mathcal{O}_\mathbb{P}$-module since $\mathcal{I}$ is principal and generated by a non-zero divisor. Thus we can tensor the Euler sequence with $\mathcal{O}_Y$ and get

$$0 \to \Omega_\mathbb{P} \otimes \mathcal{O}_Y \to \mathcal{O}_Y(-1)^5 \to \mathcal{O}_Y \to 0,$$

from which it easily follows that $H^1(Y, \Omega_\mathbb{P} \otimes \mathcal{O}_Y) \cong H^0(\mathcal{O}_Y) = k$. We conclude that $h^{11} = 1$.

Now we compute $h^{12} = \dim_k H^1(Y, \Omega^2_Y)$. This is equal to $H^2(Y, \Omega_Y)$ by Serre duality. Again we use the conormal sequence. From the Euler sequence we get that $H^2(Y, \Omega_\mathbb{P} \otimes \mathcal{O}_Y) = 0$. We also get that $h^3(Y, \Omega_\mathbb{P} \otimes \mathcal{O}_Y) = 24$. NOW $H^3(\Omega_Y) = 0$ (WHY??), and it follows from the above computations that $h^{12} = 125 - 24 = 101$.

This example is extremely important in mirror symmetry.

References


