Ordering finite labeled trees

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Abstract. Previously — in CiE 2005 [2005] — we have given an ordering of finite trees and showed that it is a well ordering reaching up to the small Veblen ordinal. Here we extend this ordering to finite labeled trees — similar to Takeutis ordinal diagrams [1975] — and show that this extended ordering is also a well ordering.

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1

We have given a wellordered set of labels Λ and consider finite trees with labels at the nodes. In the trees the branches are ordered — from left to right. Below there are four examples of finite labeled trees

Here the smallest label is 0, then 1, 2 .... The first tree is the ordinal ϵ_0. The second tree — consisting of only one node with label 1 — is the smallest tree larger than all trees with just 0 as label. It corresponds to the small Veblen ordinal. The third tree corresponds to the Howard ordinal. The fourth tree is larger than the three others and is just an example of the more general type of labeled trees considered.

2

In CiE 2005 [2005] we gave an ordering of finite trees. The ordering was given by

$$\boxed{\mathbf{A} < \mathbf{B} \;\; \Leftrightarrow \;\; \mathbf{A} \leq \langle \mathbf{B} \rangle \;\vee\; (\langle \mathbf{A} \rangle < \mathbf{B} \wedge \langle \mathbf{A} \rangle < \langle \mathbf{B} \rangle)}$$

where

$\mathbf{A} \leq \langle \mathbf{B} \rangle$: There is an immediate subtree $\mathbf{B}_i$ of $\mathbf{B}$ such that either $\mathbf{A} < \mathbf{B}_i$ or $\mathbf{A} = \mathbf{B}_i$

$\langle \mathbf{A} \rangle < \mathbf{B}$: For all immediate subtrees $\mathbf{A}_j$ of $\mathbf{A}$ we have $\mathbf{A}_j < \mathbf{B}$

$\langle \mathbf{A} \rangle < \langle \mathbf{B} \rangle$: The inverse lexicographical ordering of the immediate subtrees — we first check which sequence have smallest length, and if they have equal length we look at the rightmost immediate subtree where they differ

We then prove by induction over subtrees

- The relation is transitive
- The relation is total: $A < B \vee A = B \vee B < A$ where $A = B$ means the ordinary equality between trees and the three cases are mutually exclusive
- The relation is decidable

We used $\Pi^1_1 - CA$ to prove that the relation is a well order. There is a 1-1 correspondence between the finite trees and the ordinals less than the small Veblen ordinal.

3

In the generalization to finite labeled trees we shall

- Consider more orderings — an ordering $<_n$ for each label n and an ordering $<_\infty$
- Have a proof of the well ordering going beyond $\Pi^1_1 - CA$
- Consider a new notion — visibility in a labeled tree

First the new notion — visibility. For each label n we define n-visibility. Say that we are in a labeled tree S and at node ν. Then from node ν a node μ is n-visible if

- Node μ has label $\geq n$
- All nodes strictly between ν and μ have labels $> n$

So from node ν we can n-see all nodes which can be reached through nodes with labels $\geq n$ and up to the first nodes with label n. A node with label $\leq n$ will block the n-visibility for all nodes above it. We use the visibility to define the n-subtrees of a tree S

$$\langle S \rangle_n = \text{the sequence of all } n\text{-subtrees } n\text{-visible from the root of } S$$

Now we are ready to define the orderings

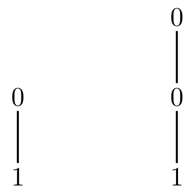
$$S <_j T \Leftrightarrow S \leq_j \langle T \rangle_j \vee (\langle S \rangle_j <_j T \wedge S <_{j+} T)$$
$$S <_\infty T \Leftrightarrow \textit{lexicographical ordering}$$

Here we use abbreviations like for the finite trees

– $\leq_j$ means either ordinary $=$ or $<_j$
– $S \leq_j \langle T \rangle_j$: There exists a j-subtree T_0 of T with $S \leq_j T$
– $\langle S \rangle_j < T$: For all j-subtrees S_0 of S , $S_0 <_j T$
– $j+$ is the smallest label in S and T larger than j if it exists, else it is ∞
– The lexicographical ordering is such that we compare in priority
  • The labels at the root of S and the root of T
  • For the same label i: The lengths of $\langle S \rangle_i$ and $\langle T \rangle_i$
  • The rightmost place where the two sequences differ in the $>_i$-ordering

As for the finite trees we get that the orderings are transitive, total and decidable. This is proved by induction over the subtrees and using the finite set of labels occurring in the trees. The following differs from our theory of finite trees

– We consider many orderings
– In the theory of finite trees to each ordinal there corresponded a unique finite tree. This is not so any more. The two finite labeled trees below is equal in the ordering



Both trees have the same 1-immediate subtrees — they have none, and to get equality this is what we compare.

4

To make the proof of the well ordering more perspicuous we go through the following cases

– Only label 0
– Finite number of labels
– A well ordered set of labels

Trees with only label 0 is the same as finite trees without labels. Our proof here is not the same as in [2005]. The proof is a variant of the usual minimal bad argument. We remind the reader

Bad tree: A tree is bad if there is an infinitely descending sequence starting with the tree
Minimal tree: A tree is minimal if no immediate subtree is bad

Now consider the case where the only label is 0. We must prove that there are no bad trees. Assume we have a bad tree S_0. If S_0 is not minimal, then by going to an immediate subtree we get a smaller tree which is bad. And then we can continue this going to immediate subtrees until we get a bad tree where all its immediate subtrees are not bad. We call this minimal bad tree for

$$S_0^0$$

Consider now the bad sequence starting with it. We shall construct a sequence of minimal bad trees. Say we have constructed the minimal bad trees

$$S_0^0 >_0 S_1^0 >_0 \cdots >_0 S_n^0$$

The sequence can be continued with bad trees

$$S_0^0 >_0 S_1^0 >_0 \cdots >_0 S_n^0 >_0 T_{n+1} >_0 T_{n+2} >_0 \cdots$$

Observe now that for any immediate subtree U_{n+1} of T_{n+1} we have $T_{n+1} >_0 U_{n+1}$ and by transitivity $S_n^0 >_0 U_{n+1}$. Hence by continuing taking immediate subtrees of T_{n+1} we get a minimal bad tree S_{n+1}^0 with

$$S_0^0 >_0 S_1^0 >_0 \cdots >_0 S_n^0 >_0 S_{n+1}^0$$

and we can continue the construction to get an infinite sequence of minimal bad trees

$$S_0^0 >_0 S_1^0 >_0 \cdots >_0 S_n^0 >_0 S_{n+1}^0 >_0 S_{n+2}^0 >_0 \cdots$$

We next observe that from the definition of the ordering

$$\boxed{S <_j T \Leftrightarrow S \leq_j \langle T \rangle_j \vee (\langle S \rangle_j <_j T \wedge S <_{j+} T)}$$

we cannot use the first condition. If $\langle S_m^0 \rangle \geq_0 S_{m+1}^0$ then S_m^0 would have an immediate subtree which is bad — contradicting the minimality of S_m^0. We must use the second condition and get

$$S_0^0 >_\infty S_1^0 >_\infty \cdots >_0 S_n^0 >_\infty S_{n+1}^0 >_\infty S_{n+2}^0 >_\infty \cdots$$

Now look at the lexicographical ordering. Here our trees have only labels 0. From some stage off all lengths of sequences of immediate subtrees must be the same, and we get an immediate subtree which starts an infinitely $>_0$-descending sequence and a bad immediate subtree — contradicting that we had a sequence of minimal trees.

5

Now we come to the case where we have a finite number of labels. We must generalize the notions of bad and of minimal

n-bad: There is an infinite $>_n$-descending sequence
n-minimal: No m-immediate subtree is m-bad for any $m \leq n$

Now assume we have a 0-bad tree. As in the previous section we construct a 0-minimal 0-bad sequence

$$S_0^0 >_0 S_1^0 >_0 S_2^0 >_0 \cdots$$

We get as above

$$S_0^0 >_1 S_1^0 >_1 S_2^0 >_1 \cdots$$

This is a 1-bad sequence. We must construct a sequence which is also 1-minimal. Observe that going to the 1-immediate subtrees does not destroy the 0-minimality. By going to the 1-immediate subtrees we may get rid of some 0-immediate subtrees but we do not create any new 0-immediate subtrees. Therefore we get a 1-minimal 1-bad sequence

$$S_0^1 >_1 S_1^1 >_1 S_2^1 >_1 \cdots$$

And then we continue this line for line until we have the sequence

$$S_0^\infty >_\infty S_1^\infty >_\infty S_2^\infty >_\infty \cdots$$

which is k-minimal for all k. Then from some stage in the sequence we have the same label n at the root and the same length of the n-immediate subtrees. But then we get an n-immediate subtree which is n-bad — contradicting the n-minimality of all trees in the sequence.

6

In the first construction of minimal bad sequence we used Π^1_1-CA. We had to decide whether an immediate subtree is bad or not. This is more problematic for the case where we have more labels. The construction of a 0-minimal 0-bad sequence of trees use Π^1_1-CA as before. But we had to do more in the construction of a 1-minimal 1-bad sequence. So say we have the 0-minimal 0-bad sequence

$$S_0^0 >_0 S_1^0 >_0 S_2^0 >_0 \cdots$$

And we get from the 0-minimality

$$S_0^0 >_1 S_1^0 >_1 S_2^0 >_1 \cdots$$

Now we had to show that this sequence can also be made 1-minimal. But then we had to decide whether an immediate subtree is starting an infinite sequence of $>_1$ descending 0-minimal trees. This goes beyond Π^1_1-CA. Further down in the construction we do the same — we had to decide whether an immediate subtree is starting an infinite sequence of $>_{n+1}$ descending n-minimal trees.

7

We now consider the general case where the labels are taken from a well ordered set. As before we assume we have a 0-bad tree, and then construct trees

$$S^n_m \text{ for each label } n \text{ and each number } m$$

Each row is a sequence of n-minimal n-bad trees

$$S^n_0 >_n S^n_1 >_n S^n_2 >_n \cdots$$

The construction goes row for row. We must say how the construction goes at limit rows. Remember that in the construction row n and $n+1$ may be quite similar. *The first place m where they differ S^{n+1}_m is a subtree of S^n_m.* We then prove that in the columns the trees are mostly as the tree above — in each column there are only a finite number of changes. For assume not. Call the leftmost column m with an infinite number of changes for the critical column. Then from some row n there are no changes to the left of the critical column m. That means that from row n all changes in the critical column comes from going from a tree to a subtree. But this can only be done a finite number of times — and the critical column was not critical. In each column there are only a finite number of changes. And we have the obvious construction for limit rows — just take the limit along each column.

Now look at a limit row

$$T^\lambda_0 >_\lambda T^\lambda_1 >_\lambda T^\lambda_2 >_\lambda \cdots$$

Here we have

- The row gives a λ-bad sequence
- All elements are n-minimal for each $n < \lambda$.

Now we can use the construction above to also get a λ-bad sequence which is λ-minimal. And then we continue as before. The contradiction comes at row ∞.

Theorem 1. *If the labels Λ is well ordered, then $<_0$ is a well order.*

8

We shall now prove that all the orderings $<_n$ are well orderings. Assume we have proved it for all labels $< n$. But then

- No tree is m-bad for $m < n$
- All trees are m-minimal for $m < n$.

Assume now we have an n-bad tree. So we get an n-bad sequence

$$T_0^n >_n T_1^n >_n T_2^n >_n \cdots$$

This sequence is m-minimal for all $m < n$ since all trees are. But then we go through the construction as above and get an n-bad n-minimal sequence

$$S_0^n >_n S_1^n >_n S_2^n >_n \cdots$$

and we can continue the construction as above to get the last row

$$S_0^\infty >_\infty S_1^\infty >_\infty S_2^\infty >_\infty \cdots$$

which is ∞-bad and m-minimal for all labels m. And then we have a contradiction as above and can conclude

Theorem 2. *All orderings $<_k$ and $<_\infty$ are well orderings.*

9

We now consider trees with labels up to some finite number N. Then consider the treeclasses

- $\mathbb{T}_N$ — trees with label N at the root
- $\mathbb{T}_{N-1}$ — trees with label $N-1$ or N at the root
- $\mathbb{T}_{N-2}$ — trees with label $N-2$, $N-1$ or N at the root
- ...
- $\mathbb{S}_0$ — trees with only label 0
- $\mathbb{S}_1$ — trees with only label 0 or 1
- $\mathbb{S}_2$ — trees with only label 0, 1 or 2
- ...

We then have

Theorem 3. *The following are isomorphic orderings*

- $\mathbb{T}_N$ *and* $<_N$ — $\mathbb{S}_0$ *and* $<_0$
- $\mathbb{T}_{N-1}$ *and* $<_{N-1}$ — $\mathbb{S}_1$ *and* $<_0$
- $\mathbb{T}_{N-2}$ *and* $<_{N-2}$ — $\mathbb{S}_2$ *and* $<_0$
- ...

And we have

Theorem 4. *The one node trees*

- 1 *is the supremum of* $\mathbb{S}_0$ *under* $<_0$
- 2 *is the supremum of* $\mathbb{S}_1$ *under* $<_0$
- 3 *is the supremum of* $\mathbb{S}_2$ *under* $<_0$
- $\ldots$

10

Let us now compare our finite labeled trees with the ordinal diagrams of Takeuti. It is easiest to compare them with the variant of ordinal diagrams of finite order made by Levitz [1970]. We use the following variant of the ordinal diagrams $O(n)$

- In $O(n)$ we consider labeled finite trees which are unordered
- We have given a pair of two numbers as the labels
  - The first number is from 0 to n and is called the order
  - The second number is a natural number called the degree
- In our labeled trees we defined sequences $\langle T\rangle_k$ for each order k
- In the ordinal diagrams $O(n)$ we define similarly multisets $[T]_k$ for each order k
- The orderings $<_k$ are defined similarly for the labeled finite trees and the ordinal diagrams using the gap condition
- The orderings $<_\infty$ are defined lexicographically but this comes out differently for our ordered labeled trees and the unordered ordinal diagrams
- In the ordinal diagrams the elements of the multisets $[T]_k$ all have the same order at the root but they may have different degrees
- Then for the multisets we first compare the elements of largest degree and so on

Levitz have the connection between the finite ordinal diagrams and iterated inductive definitions

Theorem 5 (Levitz). *The ordinal diagram* $O(n)$ *gives the ordinals connected with* ID_{n-1}*. In particular the ordinal diagrams with order 0 and 1 give the ordinal below the Howard ordinal.*

On the other hand we give connections between the unordered ordinal diagrams and our ordered finite labeled trees. We can use the degrees to embed the ordered trees into the unordered trees and use the possibility of having large branchings to embed ordered trees with large degrees. We get

Theorem 6. *The finite labeled trees* $\mathbb{S}_n$ *corresponds to the ordinal diagrams* $O(n)$*. In particular the labeled tree with the single node 2 under* $<_0$ *corresponds to the Howard ordinal.*

Helmut Pfeiffer [1972] has generalized Levitz connections between ordinal diagrams and Schüttes notation system to arbitrary wellordered set of orders. As above we also get connections to finite labeled trees where the labels come from a wellordered set.

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