Trees and ordinals
PCC 2008 – Oslo

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University of Oslo

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Abstract

We give an ordering of finite trees and show that it is a linear wellorder where the equality is the usual equality of trees. This gives a 1-1 correspondence between finite trees and ordinals up to the small Veblen ordinal and connects nicely with Veblens notation system. The ordering of finite trees are extended to finite trees with labels and a gap condition. This gives a variant of Takeutis ordinal diagrams.

Why trees?

- We use ordinal notations

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The finite trees

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The natural numbers

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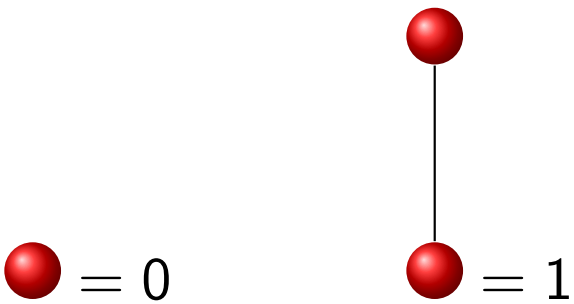

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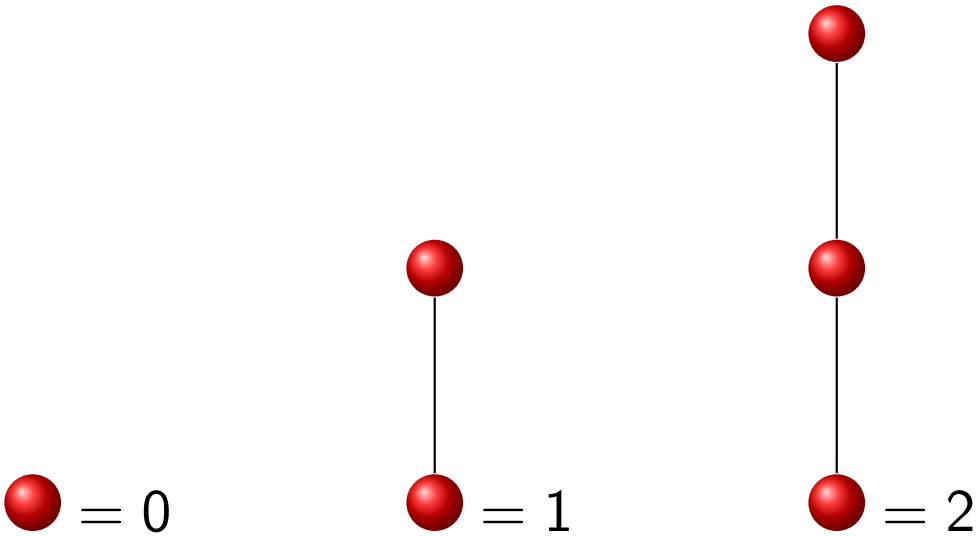

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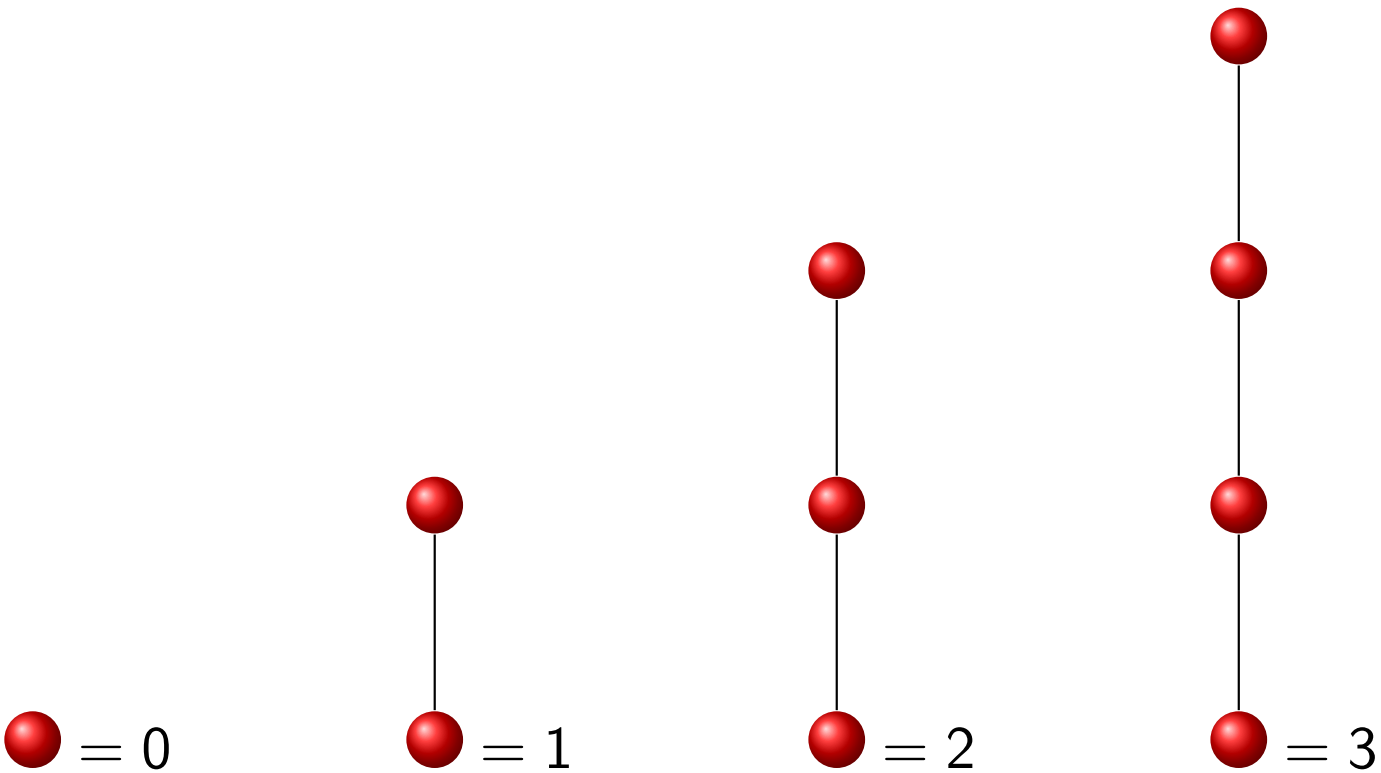

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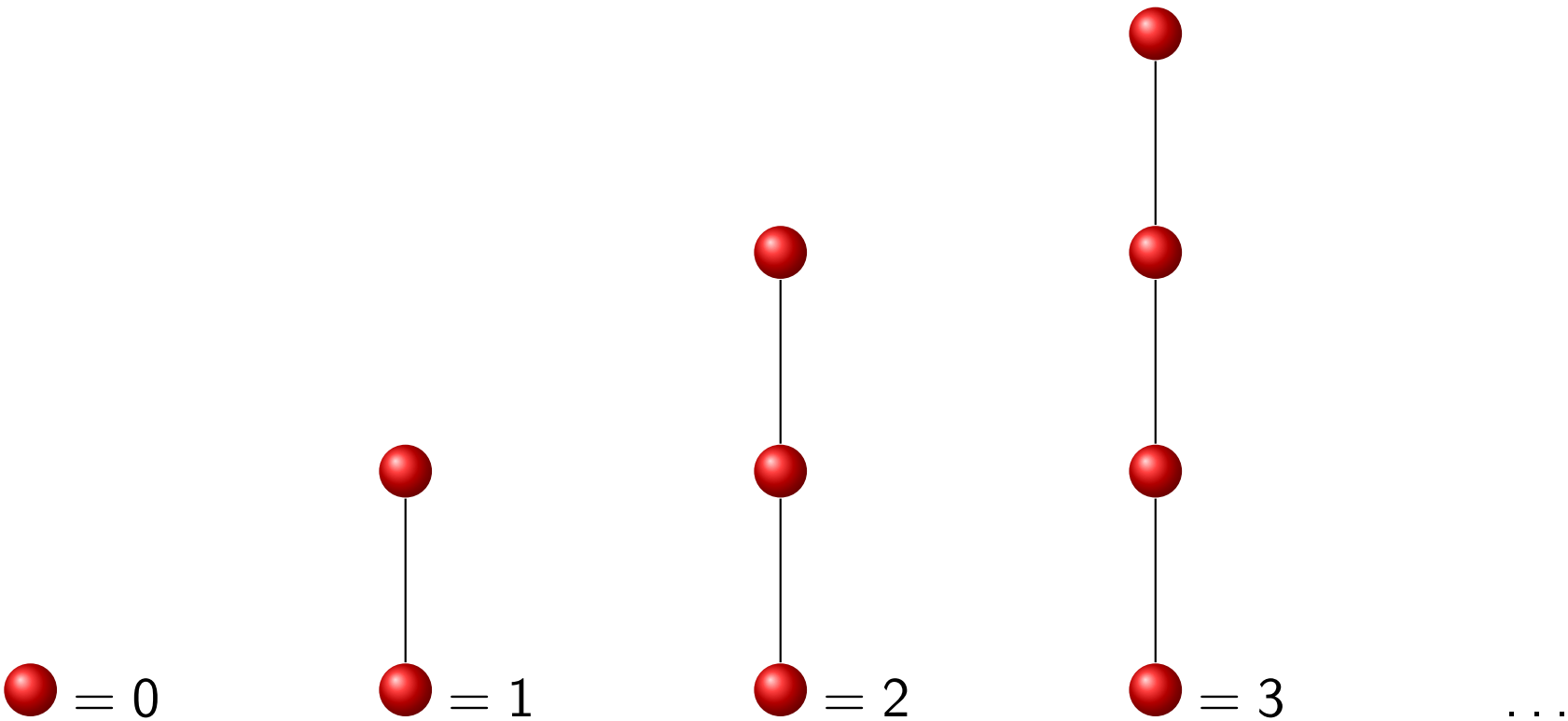

Connecting trees and ordinals

External vs internal

External Smallest ordinal a tree can be embedded into

Connecting trees and ordinals

External vs internal

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Internal Define an ordering of the trees

Connecting trees and ordinals

External vs internal

External Smallest ordinal a tree can be embedded into

Internal Define an ordering of the trees

We shall concentrate on the internal connection between trees and ordinals here.

Connecting trees and ordinals

Linear ordering of trees – definition

$$\boxed{S < T \Leftrightarrow S \leq \langle T \rangle \vee (\langle S \rangle < T \wedge \langle S \rangle < \langle T \rangle)}$$

Connecting trees and ordinals

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- $\langle S \rangle$ — sequence of immediate subtrees

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Connecting trees and ordinals

Linear orderings of trees – properties

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- Linear order

Connecting trees and ordinals

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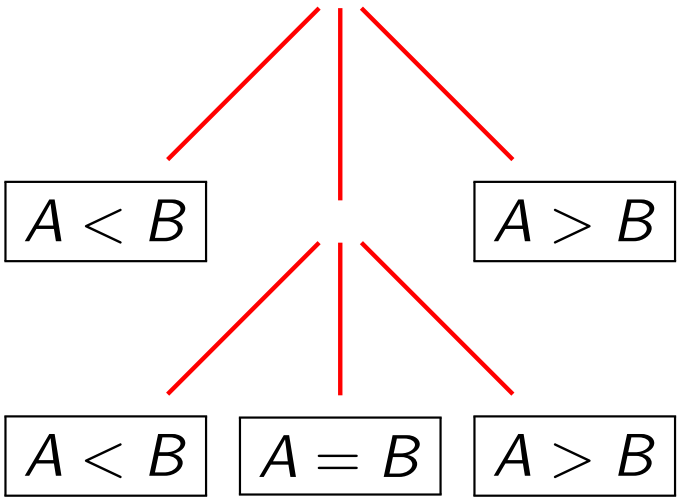
- Well order
- 1-1 correspondence with an initial segment of ordinals
- Up to the small Veblen ordinal

Decision tree

Given two trees A and B we decide the ordering by



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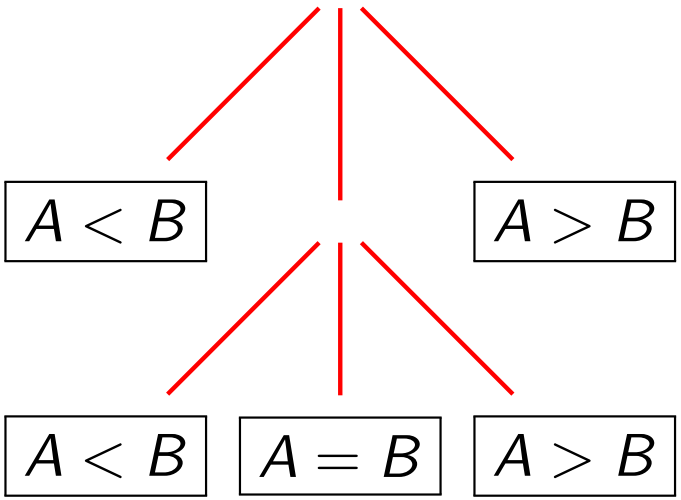

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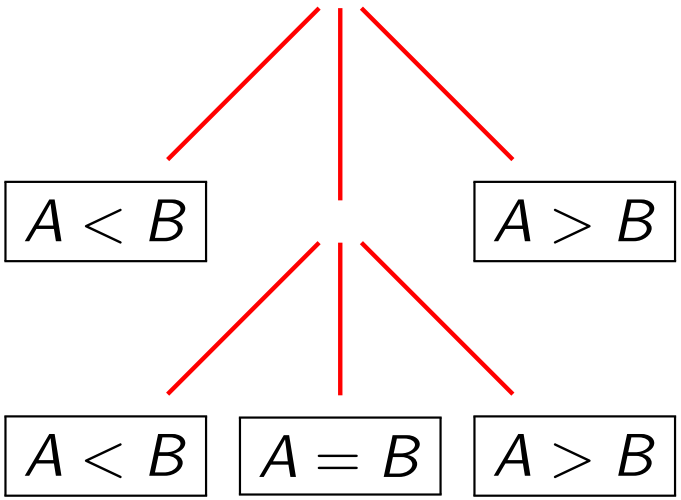

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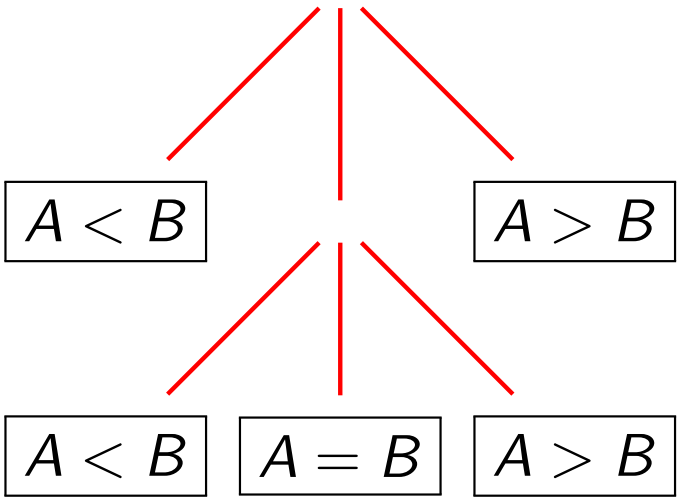

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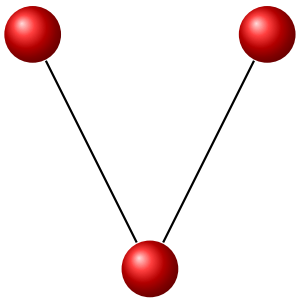
Larger ordinals



$= \omega$

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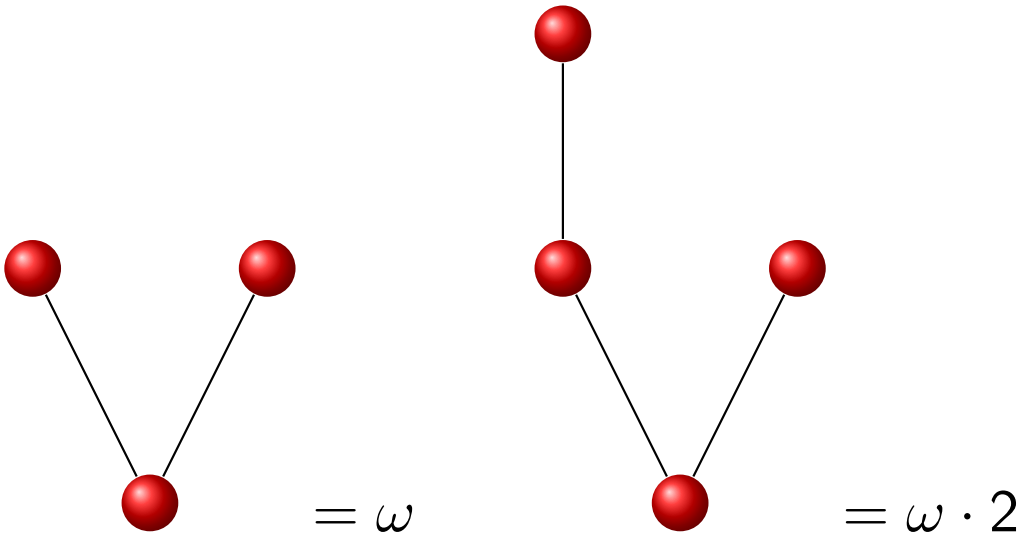

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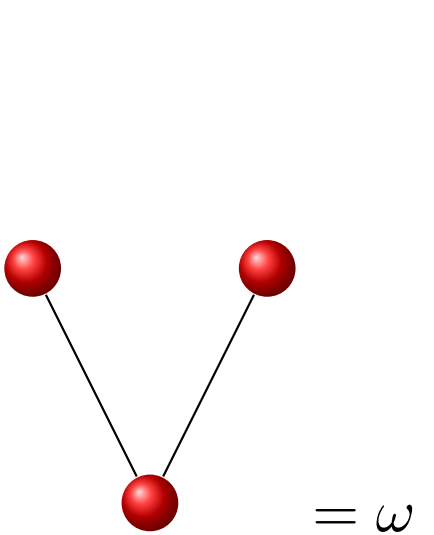

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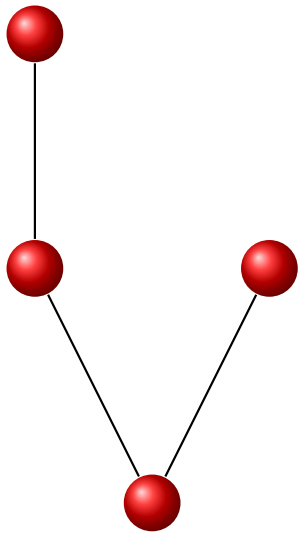
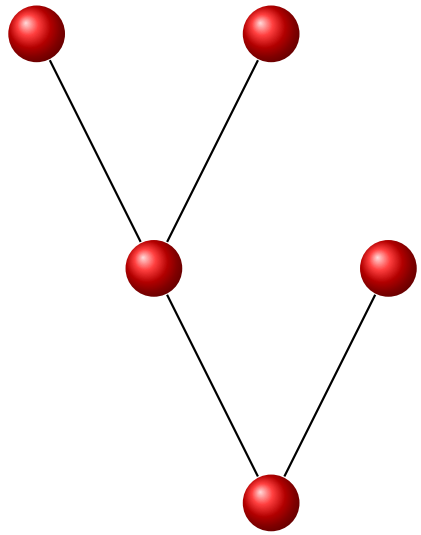
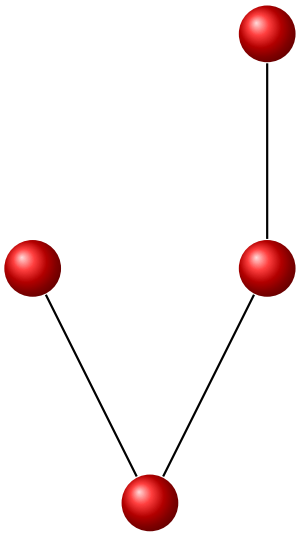


$= \omega \cdot 2$



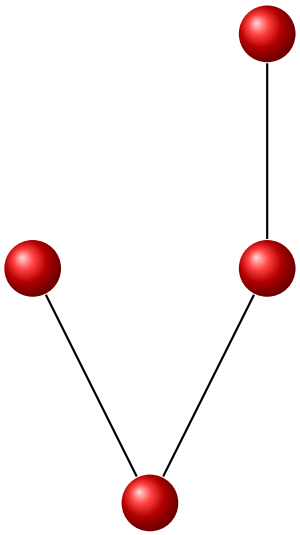
$= \omega^2$

Even larger ordinals



$= \omega^\omega$

Even larger ordinals



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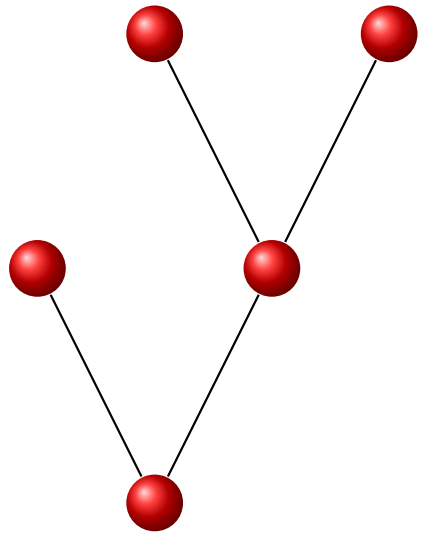

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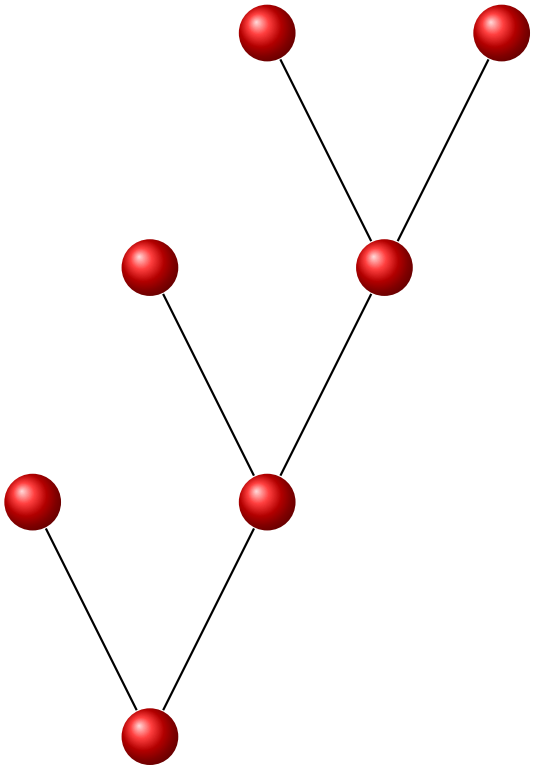

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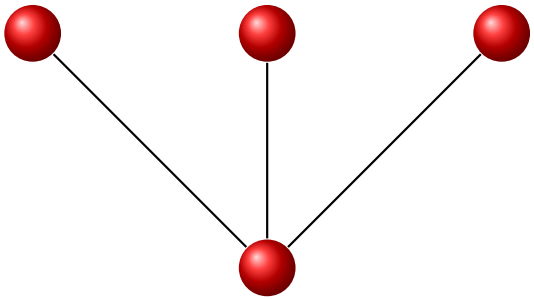
Much larger ordinals

The ϵ-numbers



$= \epsilon_0$

Much larger ordinals

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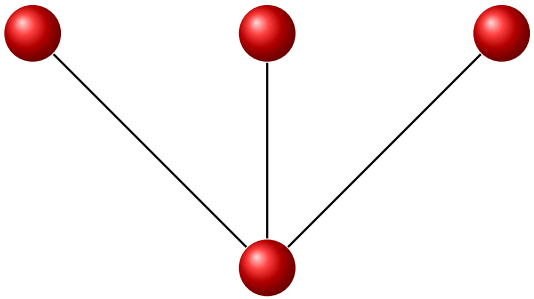

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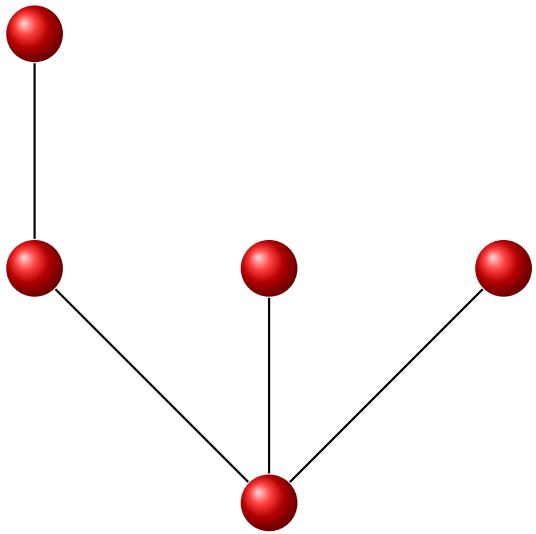

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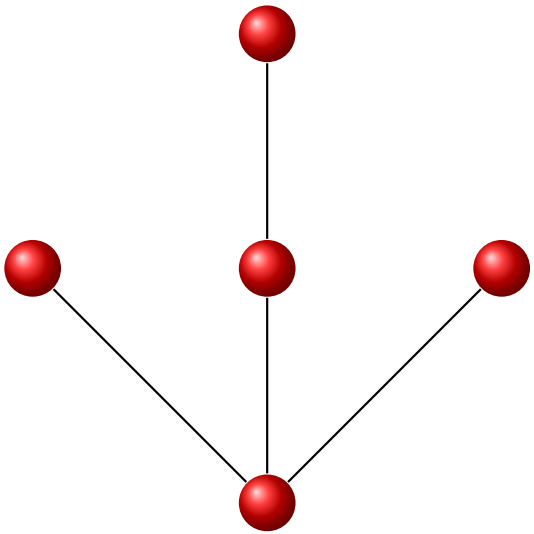
Much larger ordinals

The critical ϵ-numbers



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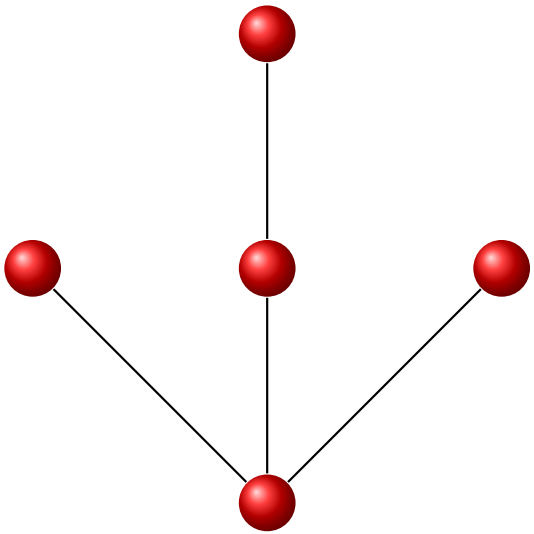

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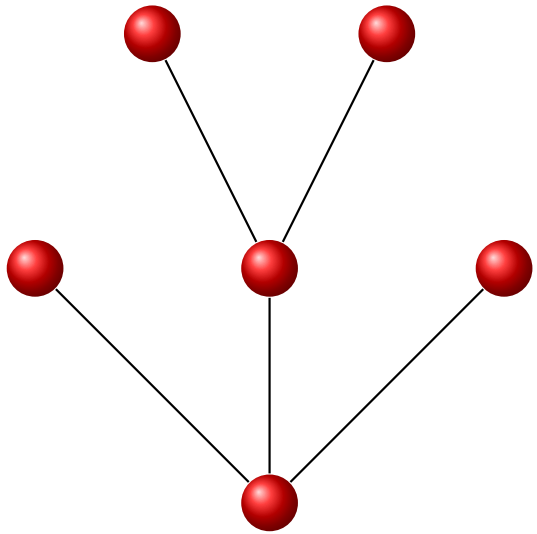

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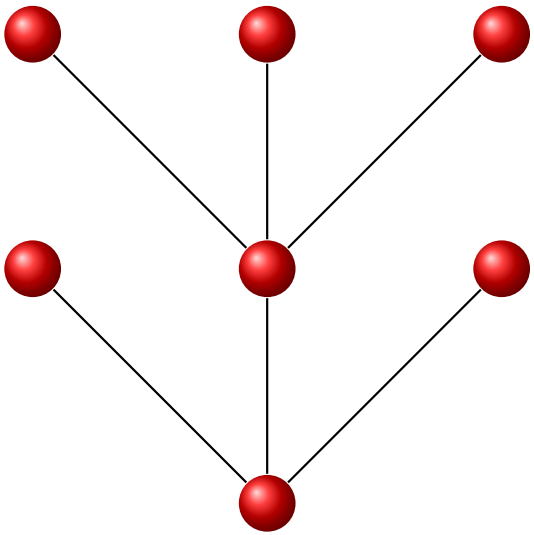

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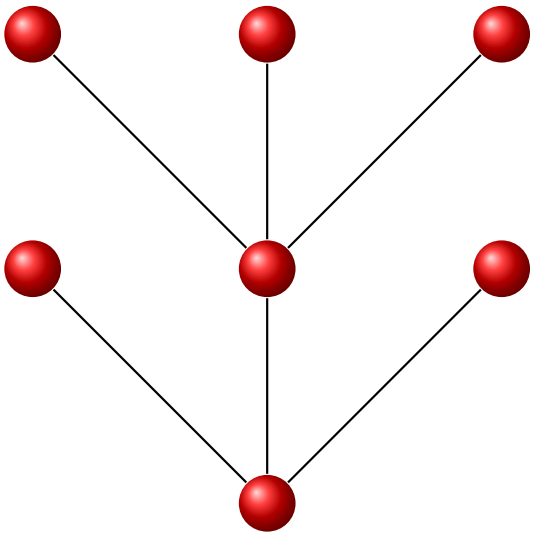
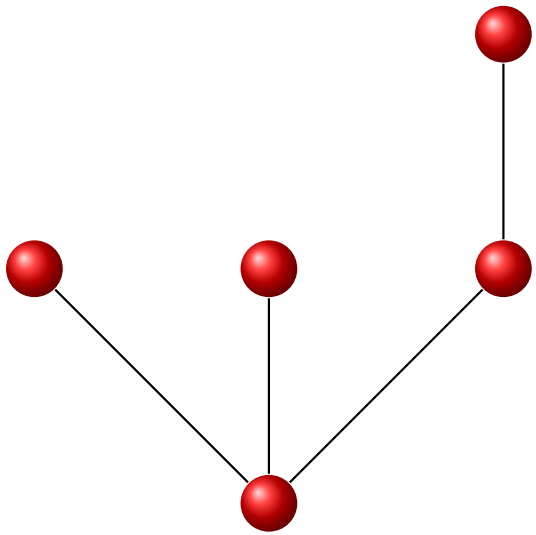
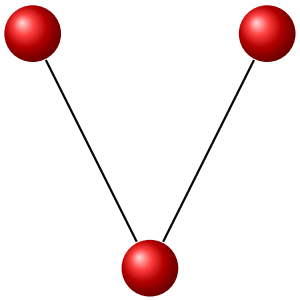


$= \kappa_{\epsilon_0}$



$= \Gamma_0$

Some important ordinals



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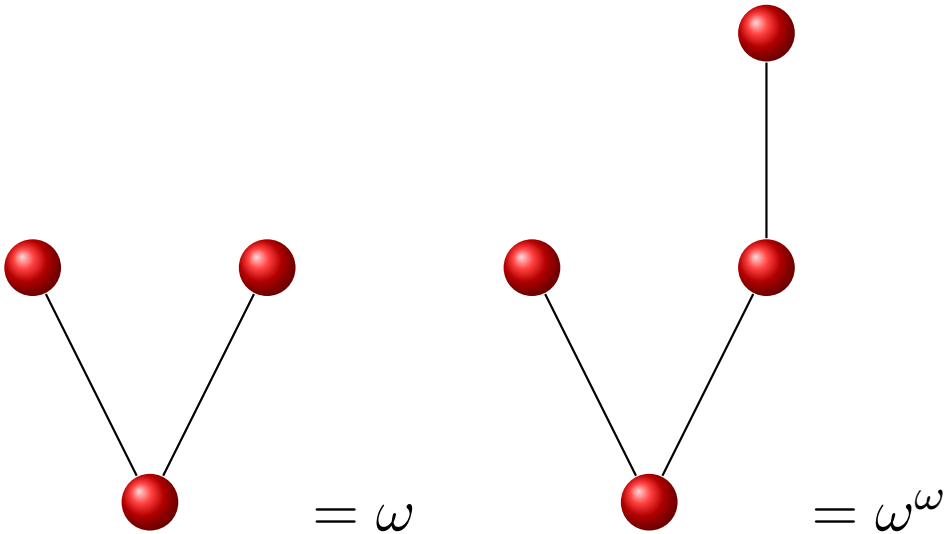

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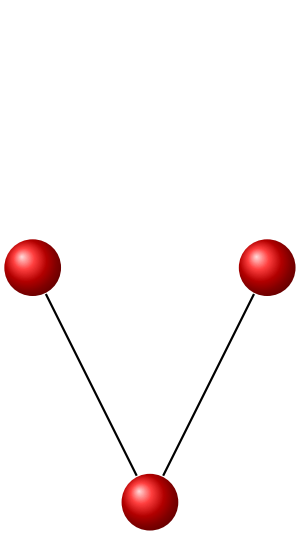

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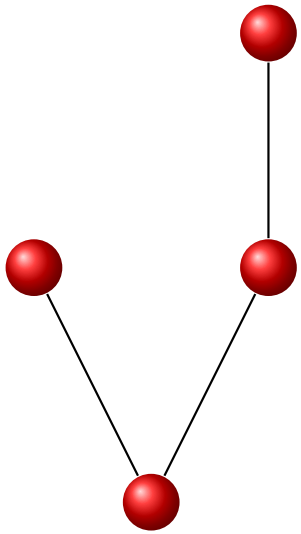

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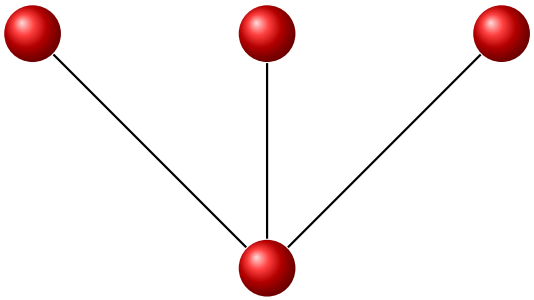

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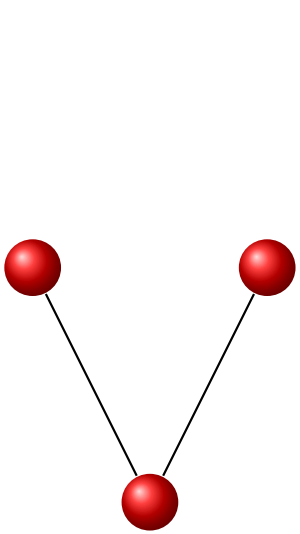

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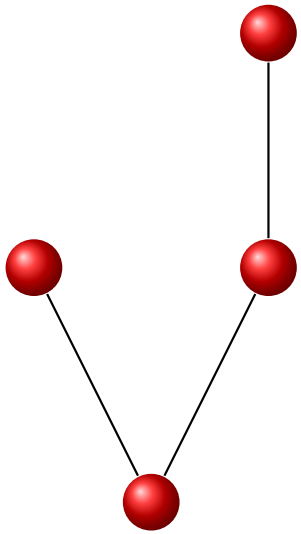

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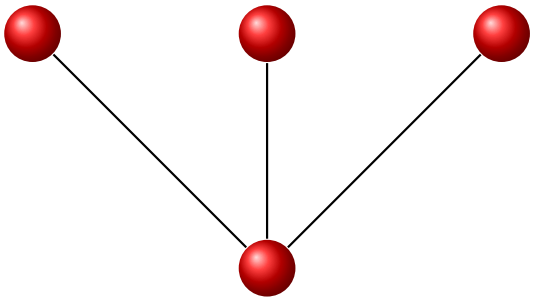

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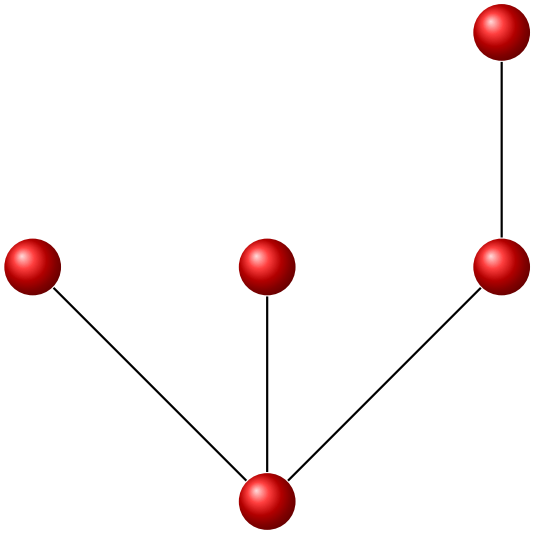


$= \Gamma_0$

Approaching a tree from below

Immediate subtrees

$$\langle A \rangle < A$$

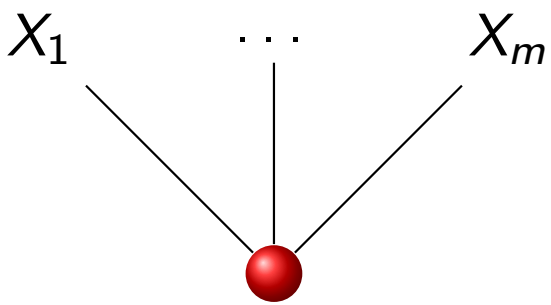
The immediate subtrees of a tree is smaller than the tree.

Approaching a tree from below

Smaller branching

Given a tree A with branching n at the root and $m < n$. Then A is closed under the function

$$X_1, \ldots, X_m \mapsto$$

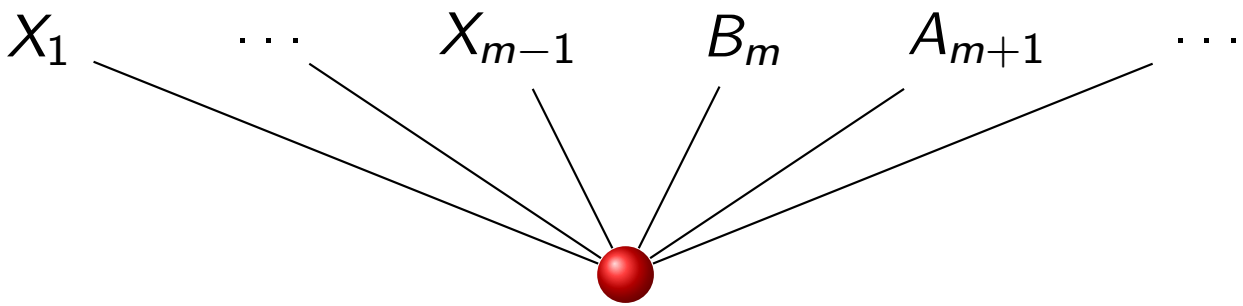


Approaching a tree from below

Lexicographical ordering

Given a tree A with immediate subtrees $A_1, \ldots, A_n$, $m \leq n$, $B_m < A_m$. Then A is closed under the function

$X_1, \ldots, X_{m-1} \mapsto$



Approaching a tree from below

Fundamental set

The fundamental set of A is the sets of all trees we get from

- Immediate subtrees

and closing under

- Smaller branching
- Less lexicographically

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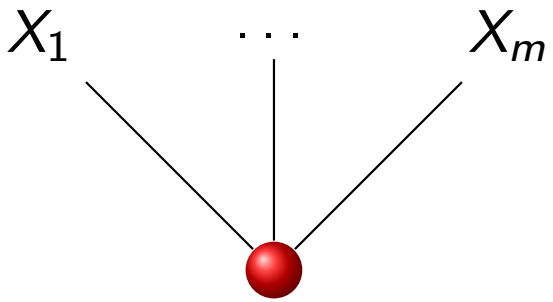
Theorem

The fundamental set of A is cofinal in A.

Well order

Using bar induction

The function

$X_1, \ldots, X_m \mapsto$ 

preserve wellfoundedness. This is proved by induction over the height of the tree and the lexicographical ordering of the sequence of wellfoundedness orderings of X_1 up to X_m.

Well order

Using minimal bad sequence

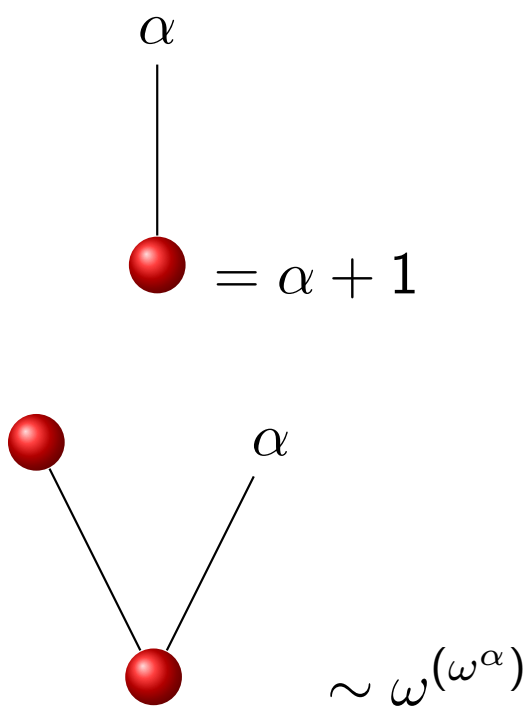
Assume there is a not well founded tree. Then we construct a minimal bad sequence $A_1 > A_2 > A_3 > \cdots$. By minimality we cannot use the condition $\langle A_i \rangle \geq A_{i+1}$ in the ordering. So we must have $\langle A_1 \rangle > \langle A_2 \rangle > \langle A_3 \rangle > \cdots$. But then by standard argument from some element of all sequences must be equally long, and then from some new element one of the elements of the sequences must be descending contradicting that our original sequence where minimal bad.

Well order

Using Kruskals theorem

Assume we have an infinite descendig sequence $A_1 > A_2 > A_3 > \cdots$, then by Kruskals theorem there is $i < j$ with A_i homeomorphically embedded in A_j. But then also $A_i \leq A_j$. Contradiction.

Some ordinal functions



Here $\sim$ indicates that we jump over the fixed points. (The tree functions do not have fixed points.)

The Veblen hierarchy

The finite trees connect up to the Veblen hierarchy — using finite number of arguments and in the enumerations we jump over fix points.

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The Veblen hierarchy

The finite trees connect up to the Veblen hierarchy — using finite number of arguments and in the enumerations we jump over fix points.

- More than two arguments
- Jump over fix points
- We may start with ordinal function ω^α
- We may have operations like natural sum

Connecting trees with ordinals

Linear extensions of homeomorphic embeddings

We consider ordering of finite trees $\prec$ which satisfies

- Linearity
- Increasing: $\langle S \rangle \prec S$
- Monotone: $\forall i. S_i \preceq T_i \Rightarrow S \preceq T$
- Branching: If branching $S >$ branching T, then $S \prec T \Rightarrow S \preceq \langle T \rangle$

Our ordering above satisfies this. But in these linear extensions there are no preference between left and right. In our lexicographical orderings the rightmost branchings are more important than the left ones.

Connecting trees with ordinals

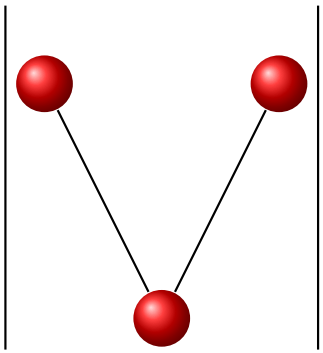
Maximal linear extensions

To each finite tree T we assign the rank

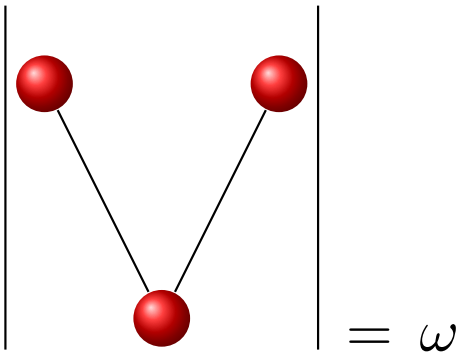
$$|T| = \sup \text{ordertype of } T \text{ in linear extensions}$$

The linear extensions are supposed to be linear extensions of homeomorphic embeddings.

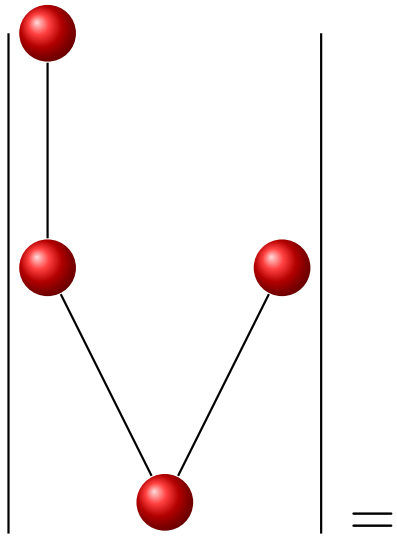
Calculating ranks


$= \omega$

Calculating ranks

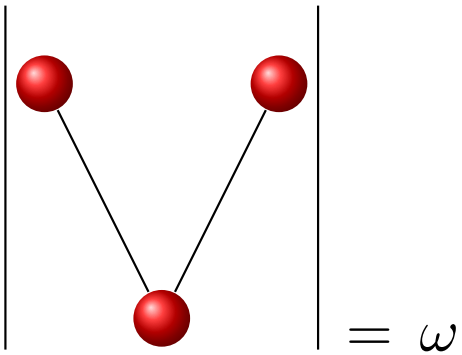


$= \omega$

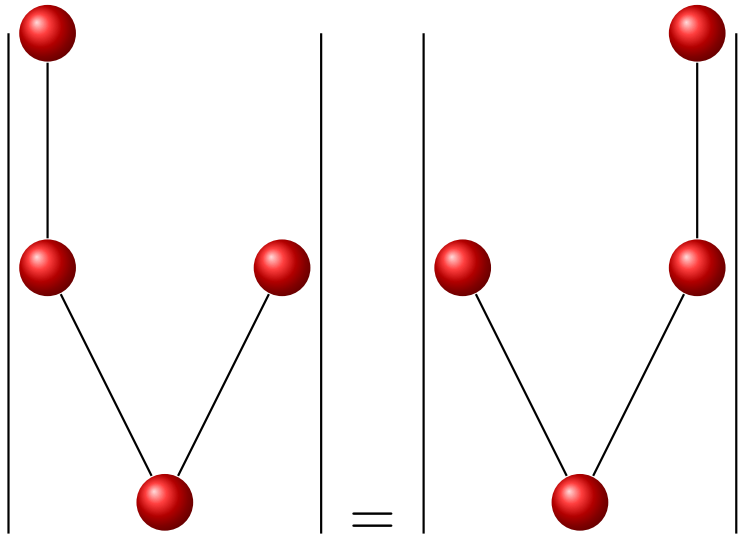


$=$

Calculating ranks

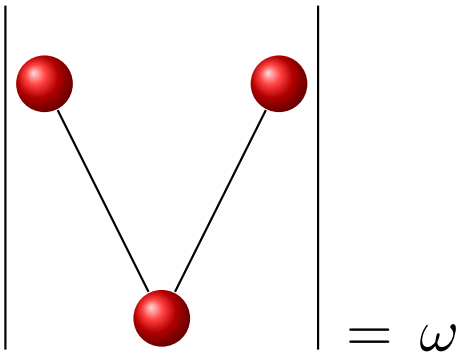


$= \omega$

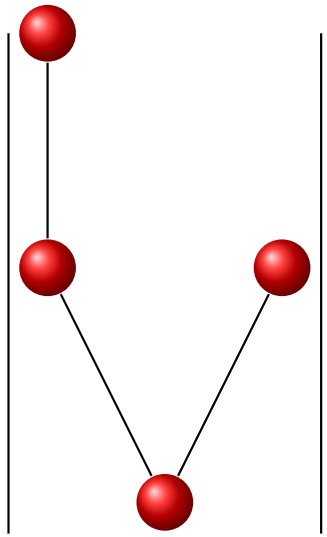


$=$

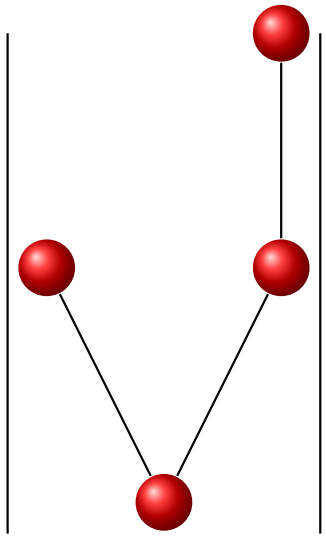
Calculating ranks



$= \omega$



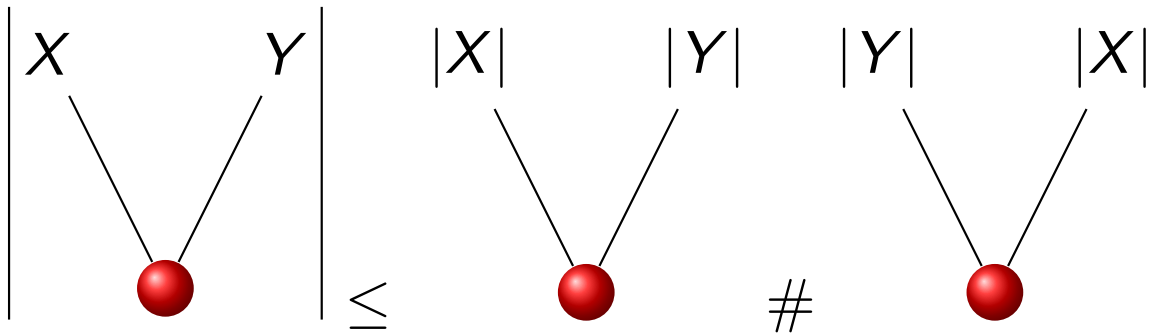
$=$



$= \omega^{\omega}$

External versus internal

For binary trees we have



In general there is a close connection between external and internal definitions of the ordertype of trees.

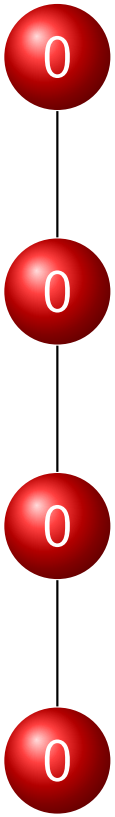
Beyond finite trees

- With the finite trees we get up to the small Veblen ordinal
- To go beyond we need finite trees with
  - labels at the nodes
  - gap condition in the embeddings

The finite labeled trees

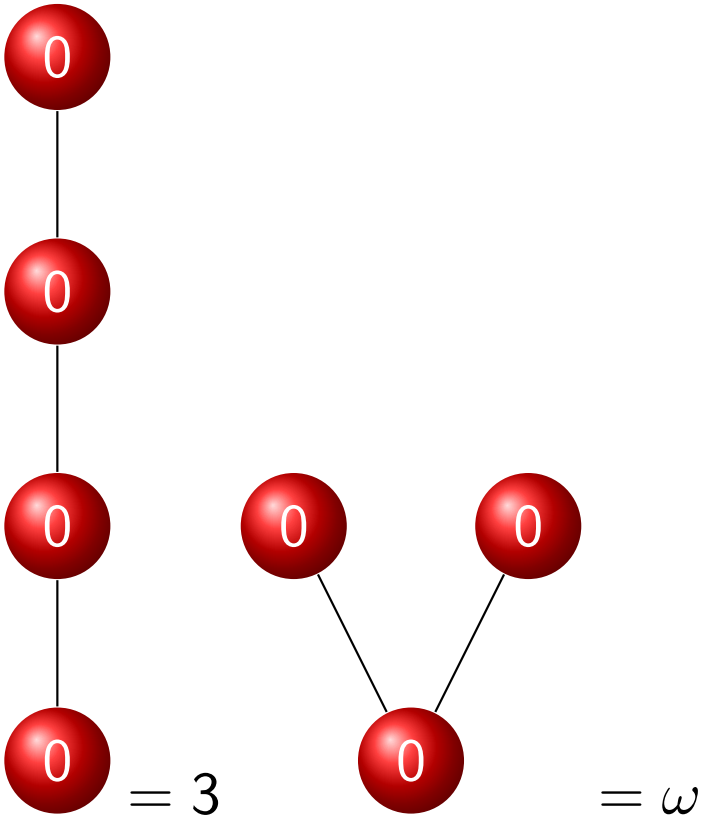
- The trees are finite
- They have root
- The branches are ordered from left to right
- Given a well ordered set of labels Λ
- Each node have a label
- The smallest label 0
- We shall give a well ordering of the trees

Small ordinals

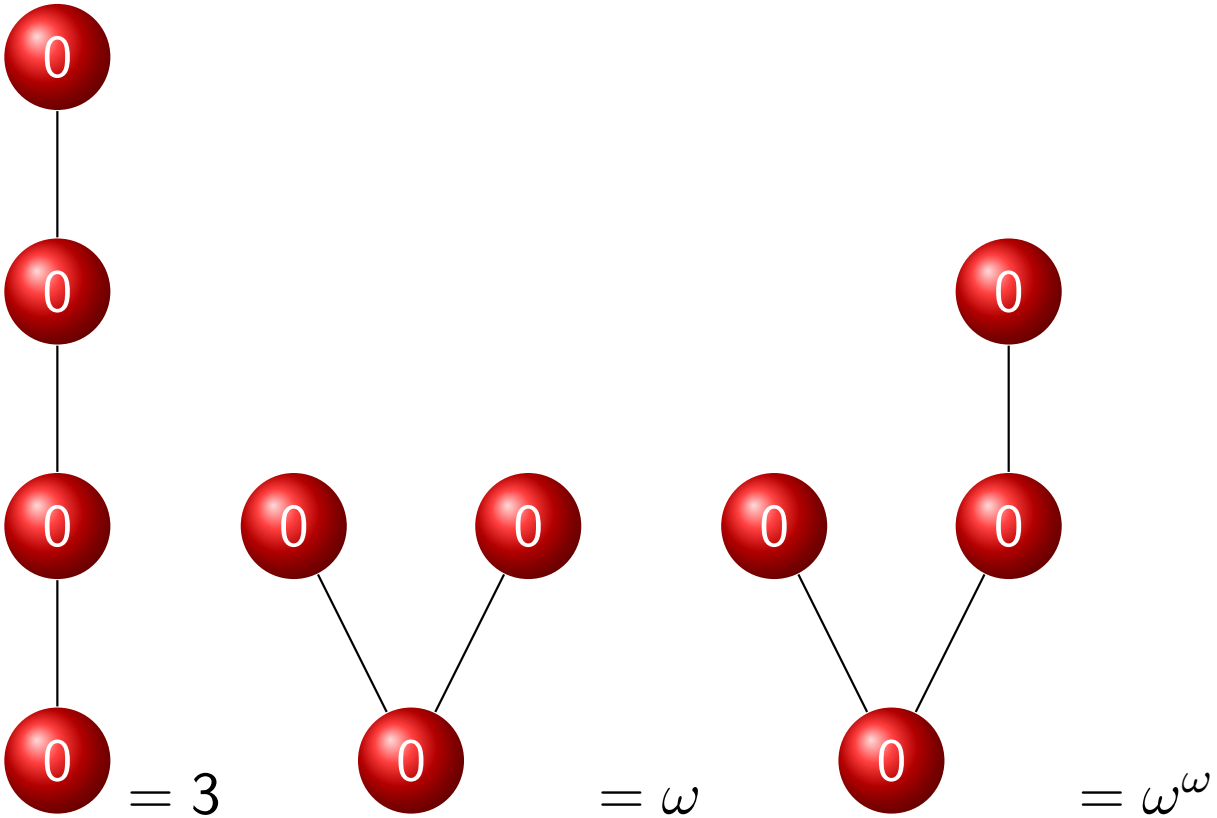


$= 3$

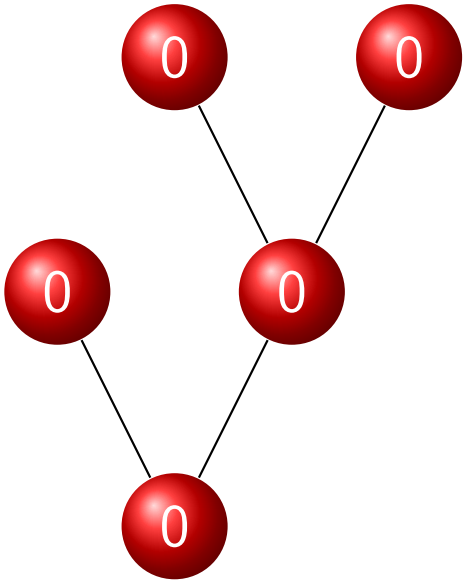
Small ordinals



Small ordinals

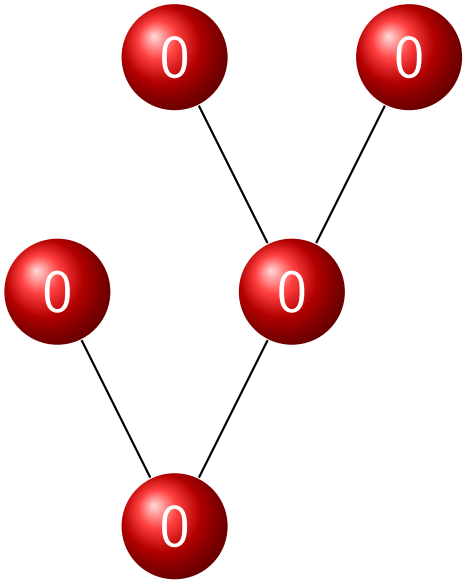


Larger ordinals

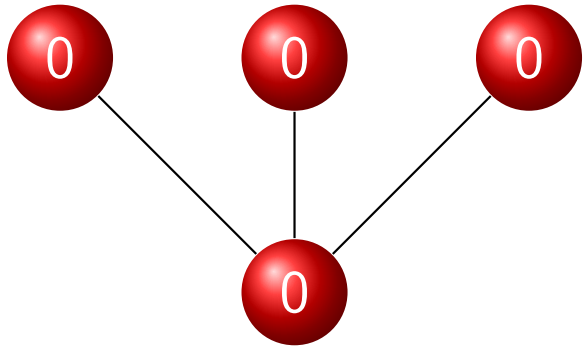


$= \omega^{\omega^\omega}$

Larger ordinals

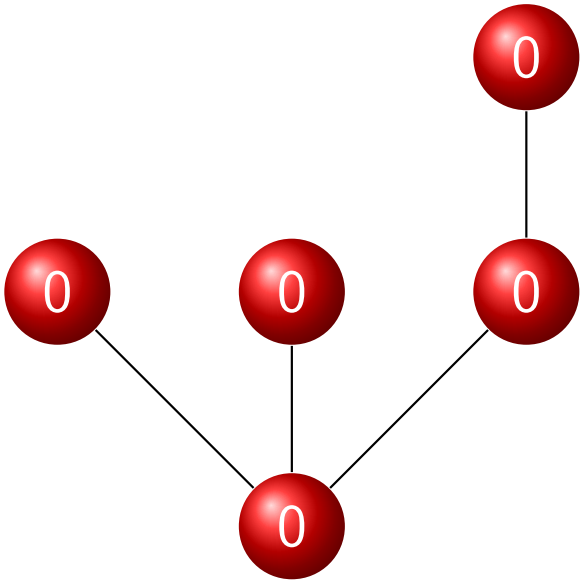


$= \omega^{\omega^\omega}$



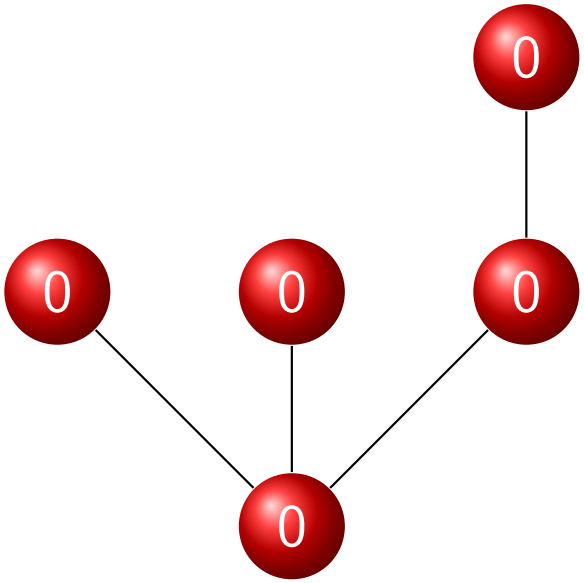
$= \epsilon_0$

Even larger ordinals



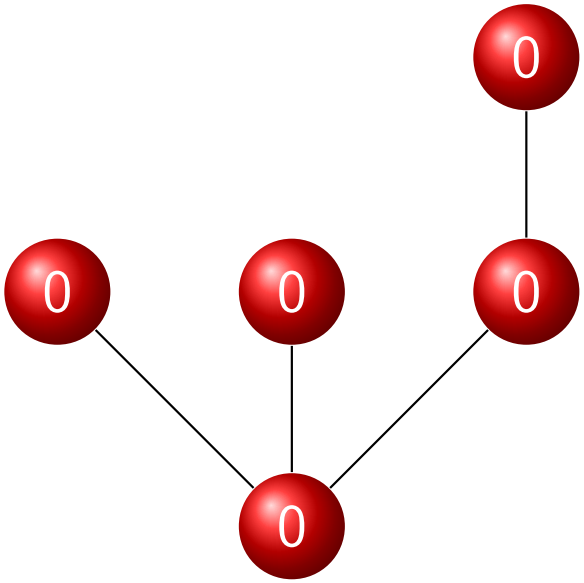
$= \Gamma_0$

Even larger ordinals



$= \Gamma_0$  $= \mathit{Veblen}$

Even larger ordinals



$= \Gamma_0$  $=$ *Veblen*  $=$ *Howard*

The ordering

The unlabeled case

$$S < T \Leftrightarrow S \leq \langle T \rangle \vee (\langle S \rangle < T \wedge \langle S \rangle < \langle T \rangle)$$

The ordering

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- $\langle S \rangle$ — sequence of immediate subtrees

The ordering

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$$S < T \Leftrightarrow S \leq \langle T \rangle \vee (\langle S \rangle < T \wedge \langle S \rangle < \langle T \rangle)$$

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- $\langle S \rangle < \langle T \rangle$ — lexicographical ordering

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  - length of sequences

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  - length of sequences
  - rightmost where they differ

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  - length of sequences
  - rightmost where they differ

The ordering is a linear wellorder. To each tree there is a unique ordinal and conversely for all ordinals less than the small Veblen ordinal.

The ordering

The labeled case

$$S <_i T \Leftrightarrow S \leq_i \langle T \rangle_i \vee (\langle S \rangle_i <_i T \wedge S <_{i+} T)$$

$$S <_\infty T \Leftrightarrow \textit{Lexicographical ordering}$$

The ordering

The labeled case

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- Define an ordering $<_i$ for each label i and $<_\infty$

The ordering

The labeled case

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$$S <_\infty T \Leftrightarrow \textit{Lexicographical ordering}$$

- Define an ordering $<_i$ for each label i and $<_\infty$
- The ordering $<_0$ is the main ordering

The ordering

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- The ordering $<_0$ is the main ordering
- Define sequence of i-immediate subtrees $\langle S \rangle_i$

The ordering

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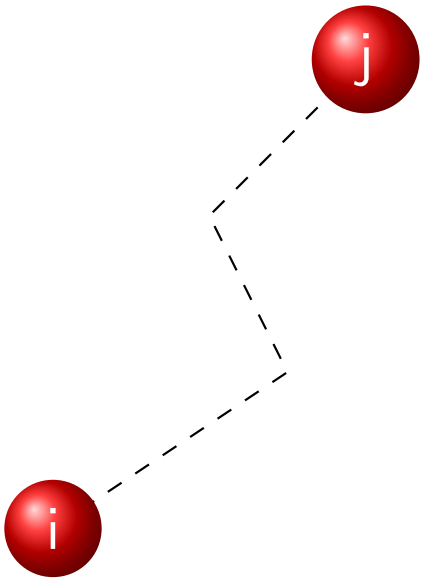
By a minimal bad argument we prove that the orderings are well orderings. This goes beyond Π^1_1.

Gaps



Assume all nodes between i and j have labels $> j$. We say that j is visible from i. In $\langle S \rangle_k$ we consider the subtrees with k at the root which are visible.

Comparison with ordinal diagrams

- Ordinal diagrams are unordered trees.
- We have defined operations like natural sum on ordinal diagrams.

The labels on the ordinal diagrams are pairs where the first component is like our labels. The second we can call degree. The degrees are necessary — we compare unordered trees with order and degree as labels with ordered trees with only order as labels.

The ordinal diagrams of a certain order and degree corresponds to our labeled trees with the same order and sufficient branching to mimic the degrees.

THE END