Finite trees as ordinals

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University of Oslo

Honouring Wilfried
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Typical trees

The natural numbers:

$$0 = \ .$$

Typical trees

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Typical trees

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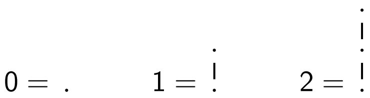

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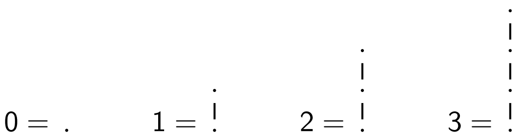

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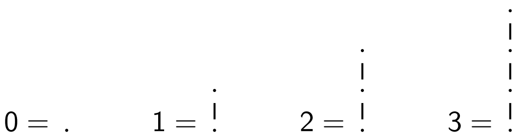

And some ordinals:

Typical trees

The natural numbers:

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$2 =$

$3 =$

And some ordinals:

$\omega =$ 

Typical trees

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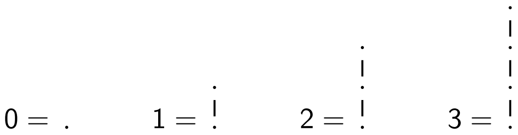

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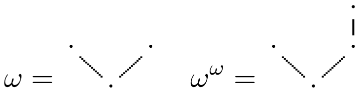

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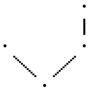
$\omega =$  $\omega^{\omega} =$  $\epsilon_0 =$ 

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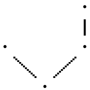
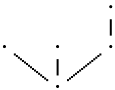
$\omega =$  $\omega^\omega =$  $\epsilon_0 =$  $\Gamma_0 =$ 

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  - Length of sequences
  - Rightmost element where they differ

Elementary properties

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- Decidable
- Transitive
- Linear
- Equality is the usual tree equality

Some ordinal functions

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 $= \alpha + 1$

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$$\sim \omega^{\omega^{\alpha}}$$

where $\sim$ means we jump over fix points.

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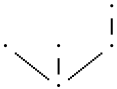
In general we get the fix point free *n*-ary Veblen functions.

Approximating from below 1

$\Gamma_0 =$ 

Start with immediate subtrees:

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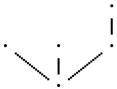
$\Gamma_0 =$ 

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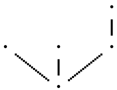
$\Gamma_0 =$ 

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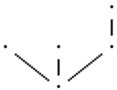
$\Gamma_0 =$ 

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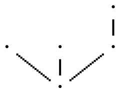
$\Gamma_0 =$ 

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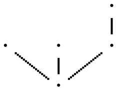

Less in lexicographical ordering:

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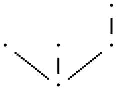
$\Gamma_0 =$ 

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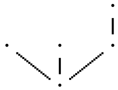
$\Gamma_0 =$ 

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This gives all trees less than Γ_0.

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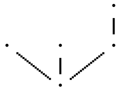
$\Gamma_0 =$ 

Less in lexicographical ordering:

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Wellfoundedness

- Minimal bad argument

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Both arguments are straightforward.

Further work

Linear extensions of embeddings

- Diana Schmidt

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- Linear extensions of topological embeddings of trees

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$$\left| \begin{matrix} A & & B \\ & \diagdown \cdot \diagup & \end{matrix} \right| \leq \begin{matrix} |A| & & |B| \\ & \diagdown \cdot \diagup & \end{matrix} \oplus \begin{matrix} |B| & & |A| \\ & \diagdown \cdot \diagup & \end{matrix}$$

This gives Higmans lemma. Further work gives Kruskals theorem.

Further work

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