Finite trees as ordinals

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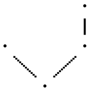
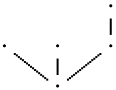
Typical trees

The natural numbers:

$0 =$ $1 =$ $2 =$ $3 =$

And some ordinals:

$\omega =$  $\omega^\omega =$  $\epsilon_0 =$  $\Gamma_0 =$ 

The ordering

$$\boxed{\mathbf{A} < \mathbf{B} \ \Leftrightarrow\ \mathbf{A} \leq \langle\mathbf{B}\rangle \ \vee\ (\langle\mathbf{A}\rangle < \mathbf{B} \wedge \langle\mathbf{A}\rangle < \langle\mathbf{B}\rangle)}$$

- $\langle\mathbf{A}\rangle$ — sequence of immediate subtrees
- $\mathbf{A} \leq \langle\mathbf{B}\rangle$ $\mathbf{A} \leq$ some immediate subtree of $\mathbf{B}$
- $\langle\mathbf{A}\rangle < \mathbf{B}$ all immediate subtrees of $\mathbf{A}$ are $< \mathbf{B}$
- $\langle\mathbf{A}\rangle < \langle\mathbf{B}\rangle$ lexicographical ordering
  - Length of sequences
  - Rightmost element where they differ

Elementary properties

$$\boxed{\mathbf{A} < \mathbf{B} \ \Leftrightarrow\ \mathbf{A} \leq \langle \mathbf{B} \rangle \ \vee\ (\langle \mathbf{A} \rangle < \mathbf{B} \wedge \langle \mathbf{A} \rangle < \langle \mathbf{B} \rangle)}$$

- Decidable
- Transitive
- Linear
- Equality is the usual tree equality

Some ordinal functions

Zero:

$$. = 0$$

Successor:

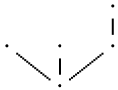
$$= \alpha + 1$$

Exponentiation:

$$\sim \omega^{\omega^{\alpha}}$$

where $\sim$ means we jump over fix points.
In general we get the fix point free *n*-ary Veblen functions.

Approximating from below 1

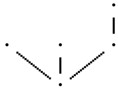
$\Gamma_0 =$ 

Start with immediate subtrees:

$0 = .$ $\quad 0 = .$ $\quad 1 =$

Use function with smaller arity:

Approximating from below 2

$\Gamma_0 =$ 

Less in lexicographical ordering:

This gives all trees less than Γ_0. To get a cofinal set we only need

Wellfoundedness

- Minimal bad argument
  - Minimal height
- Induction over wellfounded trees

Both arguments are straightforward.

Further work

Linear extensions of embeddings

- Diana Schmidt
- Linear extensions of topological embeddings of trees
- $|A|$ maximal ordertype

$$\left|\begin{array}{ccc} A & & B \\ & \diagdown\;\diagup & \\ & \cdot & \end{array}\right| \leq \begin{array}{ccc} |A| & & |B| \\ & \diagdown\;\diagup & \\ & \cdot & \end{array} \oplus \begin{array}{ccc} |B| & & |A| \\ & \diagdown\;\diagup & \\ & \cdot & \end{array}$$

This gives Higmans lemma. Further work gives Kruskals theorem.

Further work

Finite trees with labels

- Wellordered set of labels
- Each node has a label
- $\langle A \rangle_i$ – sequence of i-subtrees
- Defines $<_i$ and $<_\infty$
- $A <_i B \Leftrightarrow A \leq_i \langle B \rangle_i \vee (\langle A \rangle_i < B \wedge A <_{i+} B)$
- $A <_\infty B$ — lexicographical ordering
- Linear wellfounded preorderings
- Takeutis ordinal diagrams