Double valued reflection in the complex plane

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1 Introduction

Usually, when we talk about reflection in the plane, we think of symmetrical lines and maybe even circles. Say, reflecting over $y = 0$, we find that the point $z = (x, y)$ reflects to the point $\bar{z} = (x, -y)$. That way, a horizontal line will reflect another horizontal line on the other side of $y = 0$, and a vertical line will reflect itself.

The line $y = 0$ is defined by $r(z, \bar{z}) = z - \bar{z} = 0$, and the reflection $w$ of $z$ is defined by $r(z, \bar{w}) = z - \bar{w} = 0 \Rightarrow w = \bar{z}$.

Now, say we have a circle. What would it mean if we reflect a point over that? The idea would be to draw the normal line from the point to the circle edge and continue on the other side until you are as far away from the edge as you were before. This way, straight lines through the origin will reflect itself and a circle will permute a new circle.

The circle $|z| = R$ is defined by $r(z, \bar{z}) = z\bar{z} - R^2 = 0$ and the reflection $w$ of $z$ is defined by $r(z, \bar{w}) = z\bar{w} - R^2 = 0$, which gives $w = R^2 / \bar{z}$. See Figure 1.

![Figure 1: Reflection over the x-axis and through a circle](image)

In this report, we are going to look at what happens if we reflect over ellipses and hyperbolas in the complex plane, and consider questions like whether they are going to behave like they would in the real $xy$-plane.

All previous results and ideas are from [1].

2 The general conic

We are given that the conic with foci $\pm a$, $a > 0$ and parameter $b > 0$ is

$$|z + a| + \epsilon |z - a| = b, \quad \epsilon^2 = 1. \quad (1)$$
Which is an ellipse if \( \epsilon = 1, \ b > 2a, \) and one branch of a hyperbola if \( \epsilon = -1, \ b < 2a. \) We manipulate the expression:

\[
|z + a| + \epsilon|z - a| = b, \quad a, b \in \mathbb{R}
\]

\[
|z + a|^2 + 2\epsilon|z + a||z - a| + \epsilon^2|z - a|^2 = b^2
\]

\[
2\epsilon|z + a||z - a| = b^2 - 2(z\bar{z} + a^2)
\]

\[
4|z + a|^2|z - a|^2 = (b^2 - 2(z\bar{z} + a^2))^2
\]

\[
|z + a|^2|z - a|^2 = ((x + a)^2 + y^2)((x - a)^2 + y^2) = (x^2 - a^2)^2 + y^2(2x^2 + 2a^2) + y^4
\]

\[
= x^4 + 2x^2y^2 + y^4 + 2a^2(y^2 - x^2) + a^4
\]

\[
= (z\bar{z})^2 + a^2\left(2\left(\frac{z - \bar{z}}{2i}\right)^2 - 2\left(\frac{z + \bar{z}}{2}\right)^2 + a^2\right)
\]

\[
4|z + a|^2|z - a|^2 = (b^2 - 2(z\bar{z} + a^2))^2
\]

\[
4(z\bar{z})^2 + 4a^2 - 2a^2(z - \bar{z})^2 - 2a^2(z + \bar{z})^2 = b^4 - 4b^2(z\bar{z} + a^2) + 4(z\bar{z} + a^2)^2
\]

\[
-4a^2z\bar{z} - 4a^2\bar{z}\bar{z} + (4b^4 - 8a^2)z\bar{z} = -4a^2(z^2 + \bar{z}^2) + (4b^2 - 8a^2)z\bar{z} = b^4 - 4a^2b^2
\]

\[
\frac{4(b^2 - 2a^2)}{b^2(b^2 - 4a^2)}z\bar{z} - \frac{4a^2}{b^2(b^2 - 4a^2)}(z^2 + \bar{z}^2) - 1 = 0
\]

\[
r(z, \bar{z}) = Bz\bar{z} - A(z^2 + \bar{z}^2) - 1 = 0, \quad A = \frac{4a^2}{b^2(b^2 - 4a^2)}, \quad B = \frac{4(b^2 - 2a^2)}{b^2(b^2 - 4a^2)}
\]

The reflection \( w \) can be found by solving

\[
r(z, \bar{w}) = Bz\bar{w} - A(z^2 + \bar{w}^2) - 1 = 0
\]

As this is a quadratic equation, it will be double valued.

### 3 Transformation

By the sine transform

\[
z = a \sin t = a \sin(t' + it'')
\]

\[
= a(\sin t' \cos(it'') + \cos t' \sin(it''))
\]

\[
= a(\sin t' \cosh t'' + i \cos t' \sinh t'')
\]

we can more easily visualize the double valued reflection as it will split into two single valued ones. [2]
3.1 The ellipse

The map is a biholomorphism between the strip $|t'| < \pi/2$ and the complex plane cut from $-\infty$ to $-a$ along the negative real axis and from $a$ to $\infty$ along the negative real axis, splitting the ellipse into two parts; By transformation, the ellipse will become two horizontal lines. See Figure 2.

The following are the two reflections of $t$ over the lines $t'' = c''_0$ and $t'' = -c''_0$, defining the upper and lower halfs of the ellipse. $c''_0$ is defined below.

$$\rho_{\pm}(t', t'') = (t', \pm 2c''_0 - t'').$$

We can write the reflection $w$ as

$$w = a \sin(t' + i(\pm 2c''_0 - t''))$$
$$= a(\sin t' \cos(i(\pm 2c''_0 - t'')) \cos t' \sin(i(-2c''_0 - t'')))
$$
$$= a(\sin t' \cosh(\pm 2c''_0 - t'') + i \cos t' \sinh(\pm 2c''_0 - t''))$$

$$\Rightarrow \tilde{w} = a(\sin t' \cosh((\pm 2c''_0 - t'')) - i \cos t' \sinh((\pm 2c''_0 - t'')))
$$
$$= a(\sin t' \cos(i(\pm 2c''_0 - t'')) - \cos t' \sin(i(\pm 2c''_0 - t'')))
$$
$$= a(\sin(t' - i(\pm 2c''_0 - t'')) = a \sin(t \pm i2c''_0)

$$= a(\sin t \cos(2c''_0) \pm \cos t \sin(2c''_0))$$
$$= a(\sin t \cosh(2c''_0) \pm i \cos t \sinh(2c''_0))$$

Figure 2: Transformation of the ellipse.
Given that
\[ c''_0 = \cosh^{-1}\sqrt{\frac{B + 2A}{4A}} \Rightarrow \cosh c''_0 = \sqrt{\frac{B + 2A}{4A}} \]
we can find an expression for \( \cosh(2c''_0) \) and \( \sinh(2c''_0) \).

\[
\frac{B + 2A}{4A} = \left( \frac{4(b^2 - 2a^2)}{b^2(b^2 - 4a^2)} + \frac{2 \cdot 4a^2}{b^2(b^2 - 4a^2)} \right) \frac{b^2(b^2 - 4a^2)}{4 \cdot 4a^2}
\]
\[
= \frac{b^2}{4a^2} \Rightarrow \sqrt{\frac{B + 2A}{4A}} = \frac{b}{2a} = \cosh c''_0
\]

\[
\sinh^2 c''_0 = \cosh^2 c''_0 - 1 = \frac{b^2}{2a} - 1 \Rightarrow \frac{\sqrt{b^2 - 4a^2}}{2a} = \sinh c''_0
\]

\[
\cosh(2c''_0) = \cosh^2 c''_0 + \sinh^2 c''_0 = \frac{b^2}{4a^2} + \frac{b^2 - 4a^2}{4a^2} = \frac{b^2 - 2a^2}{2a^2}
\]

\[
\sinh(2c''_0) = 2 \sinh c''_0 \cosh c''_0 = 2 \frac{\sqrt{b^2 - 4a^2}}{2a} \frac{b}{2a} = \frac{b\sqrt{b^2 - 4a^2}}{2a^2}
\]

This can be inserted into the expression for \( \bar{w} \), yielding

\[
\bar{w} = \sin t \cdot \frac{b^2 - 2a^2}{2a} \pm i \cos t \cdot \frac{b\sqrt{b^2 - 4a^2}}{2a}
\]

We now check that this expression fits in (3).

\[
z\bar{w} = a \sin t \left( \sin t \cdot \frac{b^2 - 2a^2}{2a} \pm i \cos t \cdot \frac{b\sqrt{b^2 - 4a^2}}{2a} \right)
\]
\[
= \frac{b^2 - 2a^2}{2} \sin^2 t \pm i \frac{b\sqrt{b^2 - 4a^2}}{4a} \sin(2t)
\]

\[
z^2 = a^2 \sin^2 t
\]

\[
\bar{w}^2 = \left( \frac{b^2 - 2a^2}{2a} \right)^2 \sin^2 t \pm i \frac{b\sqrt{b^2 - 4a^2}}{4a} \cos^2 t \pm i \frac{b\sqrt{b^2 - 4a^2}}{4a} \sin(2t)
\]

\[
A(z^2 + \bar{w}^2) = \frac{1}{b^2(b^2 - 4a^2)} (4a^2 \sin^2 t + (b^2 - 2a^2)^2 \sin^2 t - b^2(b^2 - 4a^2) \cos^2 t
\]
\[
\pm ib(b^2 - 2a^2)\sqrt{b^2 - 4a^2} \sin(2t))
\]

\[
Bz\bar{w} = 2(b^2 - 2a^2) \left( \frac{b^2 - 2a^2}{2} \sin^2 t \pm i \frac{b\sqrt{b^2 - 4a^2}}{2} \sin(2t) \right)
\]
Figure 3: Transformation of the hyperbola.

\[ B \bar{z} \bar{w} - A (z^2 + \bar{w}^2) = \frac{2(b^2 - 2a^2)^2}{b^2(b^2 - 4a^2)} \sin^2 t - \frac{4a^4}{b^2(b^2 - 4a^2)} \sin^2 t - \frac{(b^2 - 2a^2)^2}{b^2(b^2 - 4a^2)} \sin^2 t + \cos^2 t \]

\[ = \frac{2(b^2 - 2a^2)^2 - 4a^4 - (b^2 - 2a^2)^2}{b^2(b^2 - 4a^2)} \sin^2 t + \cos^2 t \]

\[ = \frac{b^2(b^2 - 4a^2)}{b^2(b^2 - 4a^2)} \sin^2 t + \cos^2 t = 1 \]

This shows that

\[ B \bar{z} \bar{w} - A (z^2 + \bar{w}^2) - 1 = 0. \]

### 3.2 The hyperbola

The following are the two reflections of \( t \) over the lines \( t' = c'_0 \) and \( t' = -c'_0 \) (see Figure 3):

\[ \rho_+(t', t'') = (2c'_0 - t', t'') \quad \rho_-(t', t'') = (-2c'_0 - t', t''). \]

(5)

The only difference between the two is the sign in front of \( 2c'_0 \), that is

\[ w = a \sin(\pm 2c'_0 - t' + it''). \]
This gives
\[ \bar{w} = a \sin(\pm 2c_0' - t' - it'') = a \sin(\pm 2c_0' - t) \]
\[ = a(\sin(\pm 2c_0') \cos t - \cos(\pm 2c_0') \sin t) \]
\[ = a(\pm \sin(2c_0') \cos t - \cos(2c_0') \sin t). \]

We go on as in Section 3.1:
\[ c_0' = \sin^{-1} \sqrt{\frac{B + 2A}{4A}} \Rightarrow \sin c_0' = \sqrt{\frac{B + 2A}{4A}} = \frac{b}{2a} \]
\[ \cos^2 c_0' = 1 - \sin^2 c_0' = 1 - \frac{b^2}{4a^2} = \frac{4a^2 - b^2}{4a^2} \]
\[ \cos 2c_0' = \cos^2 c_0' - \sin^2 c_0' = \frac{2a^2 - b^2}{2a^2} \]
\[ \sin 2c_0' = 2 \sin c_0' \cos c_0' = \frac{2b\sqrt{4a^2 - b^2}}{2a^2} \]
\[ \Rightarrow \bar{w} = a \left( \pm \frac{b\sqrt{4a^2 - b^2}}{2a^2} \cos t - \frac{2a^2 - b^2}{2a^2} \sin t \right) \]
\[ = \pm \frac{b\sqrt{4a^2 - b^2}}{2a} \cos t - \frac{2a^2 - b^2}{2a} \sin t \]

And we check that this expression fits in (3):
\[ B\bar{w} = \frac{4(b^2 - 2a^2)}{b^2(b^2 - 4a^2)} a \sin t \left( \pm \frac{b\sqrt{4a^2 - b^2}}{2a} \cos t - \frac{2a^2 - b^2}{2a} \sin t \right) \]
\[ = \pm \frac{b^2 - 2a^2}{b^2(b^2 - 4a^2)} \sin 2t + \frac{2(b^2 - 2a^2)^2}{b^2(b^2 - 4a^2)} \sin^2 t \]
\[ A(z^2 + a^2) = \frac{4a^2}{b^2(b^2 - 4a^2)} \left( a^2 \sin^2 t + \left( \pm \frac{b\sqrt{4a^2 - b^2}}{2a} \cos t - \frac{2a^2 - b^2}{2a} \sin t \right)^2 \right) \]
\[ = \frac{4a^4 + (2a^2 - b^2)^2}{b^2(b^2 - 4a^2)} \sin^2 t - \cos^2 t \pm \frac{\sqrt{4a^2 - b^2} (b^2 - 2a^2)}{b(b^2 - 4a^2)} \sin 2t \]
\[ B \bar{z} \bar{w} - A(z^2 + w^2) = \frac{\pm(b^2 - 2a^2)\sqrt{4a^2 - b^2} + \sqrt{4a^2 - b^2}(b^2 - 2a^2)}{b(b^2 - 4a^2)} \sin 2t \]
\[ + \frac{2(b^2 - 2a^2)^2 - (4a^4 + (2a^2 - b^2)^2)}{b^2(b^2 - 4a^2)} \sin^2 t + \cos^2 t \]
\[ = \sin^2 t + \cos^2 t = 1 \]
\[ \Rightarrow B \bar{z} \bar{w} - A(z^2 + w^2) - 1 = 0. \]

4 Reflecting over an ellipse

The two reflections of the point \( t = t' + it'' \) are given by
\[ \rho_+(t', t'') = (t', 2c_0'' - t''), \quad \rho_-(t', t'') = (t', -2c_0'' - t''). \]

We find that the pair of segments \( t'' = \pm c'', c'' > 0 \) that transform into one confocal ellipse \( E_{c''} \), generate two ellipses by reflecting over an ellipse \( E_{c_0''} \).

4.1 Within \( E_{c_0''} \)

Say \( E_{c''} \) is inside \( E_{c_0''} \). Each point on \( E_{c''} \) is on the form \( t = t' + it'' \). By using (1) we find that the top half generates the top half of one ellipse outside \( E_{c_0''} \) and the bottom half of another ellipse, further away. Consider the values
\[ \rho_+ = 2c_0'' - c'' > 2c_0'' - c'' = c_0'', \quad \rho_- = -2c_0'' - c'' < -2c_0'' < -c_0'' \]

We can see that the distance from the positive reflection, \( \rho_+ \) to \( E_{c_0''} \) is
\[ 0 < t'' < c_0' \Rightarrow \alpha_1 = |(2c_0'' - t'') - c_0''| = |c_0'' - t''|, \]
which correspond to the distance from the original point \( t'' \) to \( c_0'' \).

The distance from the negative reflection, \( \rho_- \) to the bottom half of \( E_{c_0''} \), which is \(-c_0''\), is
\[ \alpha_2 = |(-2c_0'' - t'') - (-c_0'')| = |-c_0'' - t''| = |c_0'' + t''| \]
which correspond to the distance from the original point \( t'' \) to \(-c_0''\).

Since \( 0 < t'' < c_0'' \) we can see that the distance from \( E_{c_0''} \) to \( \rho_+ \) is less than to \( \rho_- \):
\[ \alpha_2 = |c_0'' + t''| > |c_0''| > |c_0'' - t''| = \alpha_1. \]

For the negative \( t'' \) such that \( 0 > -t'' > -c_0'' \) it will be the reverse. The positive reflection \( \rho_+ = (t', 2c_0'' - (-t'')) \) will be further away from \( E_{c_0''} \) than
the negative reflection $\rho_- = (t', -2c_0'' - (-t''))$.

For $\rho_+$:

$$\beta_1 = |(2c_0'' - (-t'')) - c_0''| = |c_0'' + t''| = \alpha_2$$

And for $\rho_-$:

$$\beta_2 = |(-2c_0'' - (-t'')) - (-c_0'')| = | - c_0'' + t''| = |c_0'' - t''| = \alpha_1$$

We see that the distance from $t''$ to its positive reflection is the same as the distance from $-t''$ to its negative reflection, so together they generate an ellipse closer to $E_{c_0''}$ than the ellipse generated by the negative reflection of $t''$ and the positive of $-t''$. See Figure 4.

4.2 **On the outside of $E_{c_0''}$**

Say that the ellipse $E_{c_0'''}$ is *outside* $E_{c_0''}$. As long as the distance from the horizontal axis to the ellipse $c_0'''$ is greater than the distance from the point $t''$ to the ellipse, $|t'' - c_0'''|$, the outcome will be similar to Section 4.1.

$$0 < c_0''' < t'' < 2c_0'''$$
We will find the positive reflection $\rho_+ = (t', 2c_0'' - t'')$ for $t'' > 0$ inside $E_{c_0''}$, with positive value:

$$2c_0'' - t'' > 2c_0'' - 2c_0'', \quad 2c_0'' - t'' < 2c_0'' - c_0'' \quad \implies 0 < 2c_0'' - t'' < c_0''.$$ 

In other words, the positive reflection will again generate the top half of an ellipse.

The negative reflection $\rho_- = (t', -2c_0'' - t'')$ will be:

$$-2c_0'' - t'' < -t'' < c_0'' < 0.$$ 

This shows that, the negative reflection $\rho_-$ will be outside the ellipse, and it will generate the bottom half of an ellipse further away from $E_{c_0''}$ than the one generated from the positive reflection $\rho_+$, which is inside. The distance from $\rho_+$ to $E_{c_0''}$ is

$$|(2c_0'' - t'') - c_0''| = |c_0'' - t''| < |c_0'' - 2c_0''| < |c_0'' + t''| = |-c_0'' - t''| = |(-2c_0'' - t'') - (-c_0'')|$$

which is the distance from $\rho_-$ to $E_{c_0''}$.

For $-t''$, the positive reflection $\rho_+$ will generate the top half of the ellipse outside, and the bottom half of the ellipse inside. See Figure 5.

### 4.3 Moving further away

Consider when $E_{c_0''}$ is outside $E_{c_0''}$, but further away from $E_{c_0''}$ than a distance of $c_0''$, but within $2c_0''$, that is $|c_0''| < |t'' - c_0''| < |2c_0''| \Rightarrow 3c_0'' > t'' > 2c_0''$. The positive reflection will then be inside $E_{c_0''}$ but will generate the bottom half of the new ellipse.

$$2c_0'' - t'' < 2c_0'' - 2c_0'', \quad 2c_0'' - t'' > 2c_0'' - 3c_0'' \Rightarrow 0 > 2c_0'' - t'' > -c_0''.$$ 

For the negative reflection:

$$-2c_0'' - t'' < -2c_0'' < -t'' < 0.$$ 

And it will also generate the bottom half of a new ellipse, but obviously a different one from the one for $\rho_+$. We use the same procedure for the reflections of $-t''$ and we see that the reflections together form two new ellipses. See Figure 6.
4.4 Moving even further away

What happens when $E_{c''}$ is further away from $E_{c_0''}$ than two distances? That is,

$$|t'' - c_0''| > 2|c_0''| \Rightarrow |t''| > 3|c_0''| \Rightarrow -t'' < -3c_0''.$$  

The positive reflection will now take on a more negative value than $-c_0'$. That is, the top half of $E_{c''}$ will give the bottom half of two new ellipses, outside of $E_{c_0''}$.

$$\rho_+ : 2c_0'' - t'' < 2c_0'' - 3c_0'' = -c_0'', \quad \rho_- : -2c_0'' - t'' < -2c_0'' - 3c_0'' = -5c_0''$$

And the opposite for the reflection of $-t''$. See Figure 7.
Figure 6: The ellipse is outside, a bit further away.

Figure 7: The ellipse is outside, far away.
4.5 The movement of reflections

$E_{c''}$ constant, and starting with $E_{c''}$ collapsed on the x-axis, the reflections of it will have the values

$$\rho_+ (c', c'' = 0) = (c', \pm 2c'_0)$$

which is the ellipse $E_{2c'_0}$ "twice".

Moving away from the x-axis, towards $E_{c''}$, the double reflections $E_{2c''}$ splits with one moving inward and the other outward. When $E_{c''} = E_{c''_0}$

$$\rho_+ (c', c''_0) = (c', c'_0), \quad \rho_-(c', c''_0) = (c', -3c'_0).$$

One reflection will be "on top" of $E_{c''}$, and the other is $E_{3c''}$. Moving away from $E_{c''}$ toward $E_{2c''}$, the reflections move further apart, and when $E_{c''} = E_{2c''_0}$

$$\rho_+ (c', c''_0) = (c', 0), \quad \rho_-(c', 2c''_0) = (c', -4c''_0).$$

One reflection collapses on the x-axis $E_0$, and the other is $E_{4c''}$.

Moving even further away, the inner reflection flips and starts moving outward again, but on the other side of the x-axis. The other reflection is still growing in size. For $E_{c''} = E_{3c''_0}$

$$\rho_+ (c', 3c''_0) = (c', -3c''_0), \quad \rho_-(c', 3c''_0) = (c', -5c''_0)$$

giving the reflections $E_{c''}$ and $E_{5c''}$.

5 Reflecting over a hyperbola

The hyperbola is transformed into two vertical lines on each side of the y-axis. We denote by $H_c$ the one hyperbola branch defined by $t' = c$.

Now, the hyperbola that functions as the "mirror", $H_{c'}$, is the two lines $c'_0$ and $-c'_0$. The reflections over these two lines are given by

$$\rho_+(t', t'') = (2c'_0 - t', t''), \quad \rho_-(t', t'') = (-2c'_0 - t', t'')$$

5.1 On the outside of $H_{c''}$

Let us look at the case when the hyperbola to be reflected, $H_{c'}$, is outside $H_{c'_0}$. That is,

$$0 < c' < c'_0.$$
Let us consider the reflection of $c'$ first. The value of the positive reflection $\rho_+$:

$$2c'_0 - c' > 2c'_0 - c'_0 = c'_0$$

And the negative reflection $\rho_-$:

$$-2c'_0 - c' < -2c'_0 < -c'_0.$$ 

In other words, $\rho_+$ lies somewhere to the right of $c'_0$ and $\rho_-$ is to the left of $-c'_0$.

The distance from $\rho_+ = (2c'_0 - c', c'')$ to $H_{c'_0}$ is

$$|(2c'_0 - c') - c'_0| = |c'_0 - c'| = \alpha_1.$$ 

The distance from $\rho_- = (-2c'_0 - c', c'')$ to $H_{c'_0}$ is

$$| -2c'_0 - c' - (-c'_0)| = |c'_0 + c'| = \alpha_2.$$ 

And, we can see that one reflection is further away than the other:

$$\alpha_1 = |c'_0 - c'| < |c'_0 + c'| = \alpha_2.$$ 

Now, reflection of the line $-c'$. The positive reflection takes on the following value

$$2c'_0 - (-c') > c'_0.$$ 

And the negative reflection

$$-2c'_0 - (-c') = -2c'_0 + c' < -2c'_0 + c'_0 = -c'_0.$$ 

Which means that $\rho_+$ lies somewhere to the right of $c'_0$ and $\rho_-$ lies to the left of $-c'_0$, and we can see that the distances correspond to the reflections of $c'$:

$$\rho_+ \rightarrow H_{c'_0}; \quad |2c'_0 - (-c') - c'_0| = |c' + c'_0| = \alpha_2$$

$$\rho_- \rightarrow H_{c'_0}; \quad | -2c'_0 - (-c') - (-c'_0)| = |c' - c'_0| = \alpha_1$$

See Figure 8.
5.2 Within $H_{c'_0}$

This is the opposite case of section 4.1, and so

$$0 < c'_0 < c'.$$

The values of the reflections when reflecting over $c'$ are

$$\rho_+ : \quad 2c'_0 - c' < 2c'_0 - c'_0 < c'_0,$$

$$\rho_- : \quad -2c'_0 - c' < -c'_0,$$

which means that $\rho_+$ is to the left of $c'_0$ and $\rho_-$ is to the left of $-c'_0$. The distance to $H_{c'_0}$ is

$$\rho_+ \rightarrow H_{c'_0}:$$

$$|2c'_0 - c' - c'_0| = |c'_0 - c'| = \beta_1,$$

$$\rho_- \rightarrow H_{c'_0}:$$

$$| -2c'_0 - c' - (-c'_0)| = |c'_0 + c'| = \beta_2,$$

and

$$\beta_1 = |c'_0 - c'| < |c'_0 + c'| = \beta_2.$$

The values of the reflections when reflecting over $-c'$ are

$$\rho_+ : \quad 2c'_0 - (-c') > 2c'_0 > c'_0,$$

$$\rho_- : \quad -2c'_0 - (-c') < -2c'_0 + c'_0 = -c'_0,$$
which means that $\rho_+$ is to the right of $c'_0$ and $\rho_-$ is to the left of $-c'_0$. The distance to $H_{c'_0}$ is

$\rho_+ \to H_{c'_0}$:

$$|2c'_0 - (-c') - c'_0| = |c'_0 + c'| = \beta_2,$$

$\rho_- \to H_{c'_0}$:

$$| -2c'_0 - (-c') - (-c'_0)| = |c' - c'_0| = \beta_1.$$

See Figure 9.

### 5.3 Exceptions

The cases described in Sections 5.1 and 5.2 are the most simple. The assumption $c'_0 < \pi/2, \ c' < \pi/2$ has been made. By transformation, the vertical line $t' = \pi/2$ would be the collapsed hyperbola; the line along the x-axis from $a$ to $\infty$. And likewise with $t' = -\pi/2$, only with $(-\infty, -a]$. If $\rho_\pm$ take on values outside the interval $[-\pi/2, \pi/2]$, the hyperbola changes its nature. The halves above and beneath the x-axis swap sides.

### 6 A pause for thought

#### 6.1 Permutations

How come ellipses permute ellipses, but keep hyperbola, while hyperbola permute hyperbola, but keep ellipses?
After using the sine transform on the conicals, ellipses and hyperbola turn into horizontal and vertical lines, respectively.

If we take a look at the expression for the reflection over an ellipse:

\[ \rho_\pm(t) = (t', \pm 2c_0'' - t'') \]

we can see that the x-coordinate stay the same, while the y-coordinate is changed. This means that the reflection of every point on a vertical line will be two points on the same line, just further up or down. A horizontal line will generate two new such lines.

The opposite will be the case for the reflection over two branches of a hyperbola

\[ \rho_+(t', t'') = (2c_0' - t', t''), \quad \rho_-(t', t'') = (-2c_0' - t', t'') . \]

Here, the y-coordinate is kept while the x-coordinate changes. So, vertical lines move, while horizontal lines stay the same.

6.2 Double valued

And why is there two reflections? Thinking back to the introduction, the difference between a circle and our ellipse is not just the shape, but also the fact that we reflect over both halfs. Splitting the circle in two, the upper and lower half, and reflecting over both, there would be two reflections in this case as well.

7 Conclusion

**Proposition.** 1. Consider the ellipse given in Section 2. We cut the x-axis from \(-\infty\) to the left focus point \(-a\), and from the right focus point \(a\) to \(\infty\). By the sine transform given in section 3, the ellipse turns into two horizontal lines in the t-plane, where \(t = t' + it''\). Reflecting over this ellipse will preserve hyperbolas but permute ellipses. There will be two values for each reflection.

2. By the sine transform, the hyperbola branch from Section 2 turns into a vertical line \(t' = c\) in the t-plane. The double reflection over this hyperbola corresponds to reflecting over this line and also the mirror line \(t' = -c\), which corresponds to the other hyperbola branch. The reflection will preserve ellipses but permute hyperbolas.
References
