ASIAN OPTIONS AND THE EFFECT OF A NON-GAUSSIAN STOCHASTIC VOLATILITY MODEL.

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Abstract. In modern asset price models, stochastic volatility plays a crucial role in order to explain several stylized facts of returns. Recently, [4] introduced a class of stochastic volatility models (the so-called BNS SV model) based on superposition of Ornstein-Uhlenbeck processes driven by subordinators. The BNS SV model forms a flexible class, where one can easily explain heavy-tails and skewness in returns and the typical time-dependency structures seen in asset return data. In this paper the effect of stochastic volatility on Asian options is studied. This is done by simulation studies of comparable models, one with and one without stochastic volatility.

INTRODUCTION

Lévy processes are popular models for stock price behavior since they allow to take into account jump risk and reproduce the implied volatility smile. Barndorff-Nielsen and Shephard [4] introduced a class of stochastic volatility models (BNS SV model) based on superposition of Ornstein-Uhlenbeck processes driven by subordinators (Lévy processes with only positive jumps and no drift). The distribution of these subordinators will be chosen such that the log-returns of asset prices will be distributed approximately normal inverse Gaussian (NIG) in stationarity. This family of distributions has proven to fit the semi-heavy tails observed in financial time series of various kinds extremely well (see Rydberg [20], or Eberlein and Keller [10]).

In the comparison of the BNS SV model, we will use an alternative model, an NIG Lévy process model (LP model), which has NIG distributed log-returns of asset prices, with the same parameters as in the BNS SV case. Unlike the BNS SV model, this model doesn’t have time-dependency of asset return data. Both models are described and the effect on pricing Asian options with the two different models will be studied. This is done by a case study with calibrated parameters on stock data of the Amsterdam-Stock-Exchange-Index (AEX). We chose Asian options because they are path dependent options. The time-dependency of the asset return data in the BNS SV model leads to a difference in pricing.

Unlike the Black-Scholes model, closed option pricing formulae are in general not available in exponential Lévy models and one must use either deterministic numerical methods (see e.g. Carr and Madan [9] for the LP model and Benth and Groth [8] for the BNS SV model) or Monte Carlo methods. In this paper we will restrict ourselves to Monte Carlo methods. As described in Benth, Groth and Kettler [7] an efficient way of simulating a NIG Lévy process is by a quasi-Monte Carlo method. We will use a simpler Monte-Carlo method, which needs bigger sample size to reduce the error. Simulating from the BNS SV model involves simulating of an Inverse Gaussian Ornstein-Uhlenbeck (IG-OU) process. The oldest algorithm of simulating a IG-OU process is described in Barndorff-Nielsen and Shephard [4]. This is a quiet bothersome algorithm, since it includes a numerical inversion of the Lévy measure of the Background driving Lévy process (BDLP). Therefore it has a large processing time, hence we will not deal with this algorithm.

The most popular algorithm is a random series representation by Rosinski [19]. The special case of the IG-OU process is described in Barndorff-Nielsen and Shephard [6]. Recently Zhang & Zhang [24] introduced an exact simulation method of an IG-OU process, using the rejection method. We will compare the efficiency and reliability of these last two algorithms.

1. THE MODELS

To price derivative securities, it is crucial to have a good model of the probability distribution of the underlying asset. The most famous continuous-time model is the celebrated Black-Scholes model...
(also called geometric Brownian motion), which uses the Normal distribution to fit the log-returns of the underlying asset. However the log-returns of most financial assets do not follow a Normal law. They are skewed and have actual kurtosis higher than that of the Normal distribution. Hence other more flexible distributions are needed. Moreover to model the behavior through time we need more flexible stochastic processes. Lévy processes have proven to be good candidates, since they preserve the property of having stationary and independent increments and they are able to represent skewness and excess kurtosis. In this section we will describe the Lévy process models we use. We also give a method how to fit the models on historical data.

1.1. Distributions. The inverse Gaussian distribution IG(δ, γ) is a distribution on \( \mathbb{R}_+ \) given in terms of its density,

\[
f_{\text{IG}}(x; \delta, \gamma) = \frac{\delta}{\sqrt{2\pi}} \exp(\delta \gamma x^2) \exp \left\{ -\frac{1}{2} \left[ (\delta^2 x^{-1} + \gamma^2 x) \right] \right\}, \quad x > 0
\]

where the parameters \( \delta \) and \( \gamma \) satisfy \( \delta > 0 \) and \( \gamma \geq 0 \). The IG distribution is infinitely divisible, self-decomposable. The associated Lévy process is a jump process of finite variation.

The normal inverse Gaussian (NIG) distribution has values on \( \mathbb{R} \) and is defined by its density function,

\[
f_{\text{NIG}}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha}{\pi} \exp \left( \delta \sqrt{\alpha^2 - \beta^2} + \beta (x - \mu) \right) \frac{K_1 \left( \delta \sqrt{1 + \left( \frac{x - \mu}{\beta} \right)^2} \right)}{\sqrt{1 + \left( \frac{x - \mu}{\beta} \right)^2}}
\]

where \( K_1 \) is the modified Bessel function of the third kind and index 1. Moreover the parameters are such that \( \mu \in \mathbb{R}, \delta \in \mathbb{R}_+ \) and \( 0 \leq \beta < |\alpha| \). The NIG distribution is infinitely divisible and there exists a NIG Lévy process.

The NIG distribution can be written as a mean-variance mixture of a normal distribution with an IG(\( \delta, \sqrt{\alpha^2 - \beta^2} \)) distribution (see Barndorff-Nielsen [1]). More specifically, if we take \( \sigma^2 \sim IG(\delta, \sqrt{\alpha^2 - \beta^2}) \) independently distributed of \( \epsilon \sim N(0, 1) \), then \( x = \mu + \beta \sigma^2 + \sigma \epsilon \) is distributed NIG(\( \alpha, \beta, \delta, \mu \)). See e.g. Barndorff-Nielsen [2] for an extensive description of the above distributions.

In this text we will work on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) equipped with a filtration \( \{\mathcal{F}_t\}_{t \in [0,T]} \) satisfying the usual conditions\(^1\), with \( T < \infty \) being the time horizon.

1.2. Exponential Lévy process model. As model without stochastic volatility we will use a fitted NIG distribution to the log-returns. i.e.

\[
d \log S(t) = dX(t),
\]

where \( S \) is an arbitrary stock-price and \( X \) is an NIG Lévy process. This model is flexible to model with, but a drawback is that the returns are assumed independently. In literature this model has also been referred to as exponential Lévy process model, since the solution is of the form,

\[
S(t) = S(0) e^{X(t)}.
\]

1.3. Barndorf-Nielsen and Shephard stochastic volatility model. In practice there is evidence for volatility clusters. There seems to be a succession of periods with high return variance and with low variance. This means that large price variations are more likely to be followed by large price variations. This observation motivates the introduction of a model where the volatility itself is stochastic.

The Barndorf-Nielsen and Shephard stochastic volatility (BNS SV) model is an extension of the Black-Scholes model, where the volatility follows an Ornstein-Uhlenbeck (OU) process driven by a subordinator.

In the Black-Scholes model the asset price process \( \{S(t)\}_{t \geq 0} \) is given as a solution to the SDE,

\[
d \log S(t) = \{ \mu + \beta \sigma^2 \} dt + \sigma dB(t)
\]

where an unusual drift is chosen and \( B(t) \) is a standard Brownian motion. The BNS SV model allows \( \sigma^2 \) to be stochastic. More precisely \( \sigma^2(t) \) is an OU process or a superposition of OU processes i.e.

\[
\sigma^2(t) = \sum_{j=1}^{n} a_j \sigma^2_j(t)
\]

\(^1\)See e.g. Protter [18]
with the weights $a_j$ all positive, summing up to one and the processes $\{\sigma_j^2(t)\}_{t \geq 0}$ are OU processes satisfying,

$$d\sigma_j^2(t) = -\lambda_j \sigma_j^2(t) dt + dZ_j(\lambda_j t)$$

where the processes $Z_j$ are independent subordinators i.e. Lévy processes with positive increments and no drift. Since $Z_j$ is a subordinator the process $\sigma_j^2$ will jump up according to $Z_j$ and decay exponentially afterwards with a positive rate $\lambda_j$. As $Z_j$ is used to drive the OU process, we shall call $Z_j(t)$ a background driving Lévy process (BDLP).

We can rewrite equation (2) into,

$$d \log S(t) = \{\mu + \beta \sigma^2(t)\} dt + \sigma(t) dB(t) := dX(t).$$

We can reformulate this into,

$$\log S(t) = \log S(0) + \mu t + \beta \sigma^2(t) + \int_0^t \sigma(s) dB(s)$$

where $\sigma^2(t)$ is the integrated process, i.e.

$$\sigma^2(t) = \int_0^t \sigma^2(u) du.$$ 

Barndorff-Nielsen and Shephard [5] showed that the tail behavior (or superposition) of an integrated IG process is approximately Inverse Gaussian. Moreover an integrated superposition of IG processes will show similar behavior as just one integrated IG process with the same mean. For $\sigma^2$ an IG($\delta, \gamma, \lambda$)-OU process, $\sigma^2(t)$ has stationary marginal law IG($\delta, \gamma$). Hence by taking the $\sigma^2(t)$ an inverse Gaussian Ornstein Uhlenbeck (IG-OU) process one gets the same tail behavior as taking $\sigma^2(t)$ a superposition of IG-OU processes. So the log returns will be approximately NIG distributed, since the NIG is a mean-variance mixture.

1.4. Parameter estimation: The stationary distribution of $\sigma^2_j$ is independent of $\lambda_j$ (see Barndorff-Nielsen and Shephard [3], Th. 6.1). This allows us to calibrate the log return distribution to the data separately of the $\lambda_j$’s.

In order to get comparable models we will assume that both models have NIG distributed returns with the same parameters. The parameters will be estimated with a maximum likelihood estimation.

The maximum-likelihood estimator $(\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\mu})$ is the parameter set that maximizes the likelihood function

$$L(\alpha, \beta, \delta, \mu) = \prod_{i=1}^n f_{NIG}(x_i; \alpha, \beta, \delta, \mu)$$

where $x_i$ are points of our data-set. Since the NIG distribution is a mean-variance mixture we can choose the parameters of the IG-OU process such that returns will be approximately NIG distributed.

Least square estimation. In Barndorff-Nielsen and Shephard [4] the analytic auto-covariance function of the squared log returns is calculated and is found to be given by,

$$\text{cov}(y_n^2, y_{n+s}^2) = \omega^2 \sum a_i \lambda_i^{-2} (1 - e^{-\lambda_i \Delta t})^2 e^{-\lambda_i \Delta t(s-1)}$$

where $\omega^2$ is equal to the variance of $\sigma^2(t)$ and $a_i$'s are the weights of the different OU processes. Looking at the empirical auto-covariance function of the squared log-returns of financial data it seems that this function fits the data really well (see Figure 1). One could see that the empirical auto-covariance function of real financial data looks like a sum of exponentially decaying functions.

We estimated the empirical auto-covariance function $\gamma(s)$ for a data-set $x_1, \ldots, x_n$ by,

$$\gamma(s) = \frac{1}{n-s} \sum_{i=1}^{n-s} \left( x_{i+s}^2 - \frac{1}{n-s} \sum_{j=s}^{n} x_j^2 \right) \left( x_i^2 - \frac{1}{n-s} \sum_{j=1}^{n-s} x_j^2 \right)$$

For the calibration we will use a non-linear least square comparison of the empirical auto-covariance function with the analytic auto-covariance function (5). Hence we will minimize,

$$\sum_s \left( \gamma(s) - \text{cov}(y_n^2, y_{n+s}^2) \right)^2$$
For a non-linear least square comparison several algorithms are available. We used a standard function of Matlab which is based on the Gauss-Newton method. Note that the least square comparison is only done on the $\lambda_i$’s and $a_i$’s since $\omega$ is already given from the maximum likelihood estimation.

2. Pricing

An Asian option is an option where the payoff is not determined by the underlying price at maturity but by the average underlying price over some pre-set period of time. We will use Asian option because it is a path dependent option. Therefore it is possible to see a difference in pricing between our two models.

Let $X(t)$ be a process with càdlàg sample paths, given as in equation (1) or (4). Consider the following exponential model for the asset price dynamics,

$$S(t) = S(0)e^{X(t)}$$  

(6)

Now the aim is to price arithmetic Asian call options written on $S(t)$ . Consider such an option with exercise at time $T$ and strike price $K$ on the average over the time span up to $T$. The risk-neutral price is (See e.g. [16]),

$$A(0) = e^{-rT}\mathbb{E}_Q\left[\max\left\{\frac{1}{T}\int_0^T S(t)dt - K, 0\right\}\right]$$

Here $r$ is the risk-free interest rate and $Q$ is an equivalent martingale measure (EMM). Unfortunately, as in most realistic models, there is no unique equivalent martingale measure; the proposed Lévy models yield an incomplete market. This means there exist an infinite number of EMM’s for both models, hence there is no unique arbitrage free price.

In the case of the exponential Lévy process model with NIG distributed increments, the Esscher transform is a good candidate since it is not restrictive on the range of viable prices and it preserves the structure, meaning that under $Q$ the distribution of the returns remains in the class NIG. In particular, if $X(t) \sim \text{NIG}(\alpha, \beta, \delta, \mu)$ then under $Q$, $X(t)$ is distributed $\text{NIG}(\alpha, \beta, \delta, \mu)$. Where

$$\hat{\beta} = \frac{1}{2} + \text{sgn}(\beta)\sqrt{\frac{(\mu - r)^2}{\delta^2 + (\mu - r)^2\alpha^2} - \frac{(\mu - r)^2}{4\delta^2}}$$

The structure of a general equivalent martingale measure for the BNS SV case and some relevant subsets are studied in Nicolato and Vernardos [17]. Of special interest is the structure preserving subset of martingale measures under which the log returns are again described by a BNS SV model, although possibly with different parameters and different stationary laws. Nicolato and Vernardos [17] argue that it suffices to consider only equivalent martingale measures of this subset. Moreover they show that the
dynamics of the log price under such an equivalent martingale measure $Q$ are given by,

$$d \log S(t) = \{r - \frac{1}{2}\sigma^2(t)\}dt + \sigma(t)dB(t)$$

$$d\sigma^2(t) = -\lambda_j \sigma^2_j(t)dt + dZ_j(\lambda_j t)$$

where \{\(B(t)\)\}_{t \geq 0} is a Brownian motion under $Q$ independent of the BDLP. We choose the stationary distribution of $\sigma^2_j$ such that the stationary distribution of the variance $\sigma^2$ are approximately equal distributed after the change of measure i.e. $\sigma^2_j \sim \text{IG}($δ, γ, λj) with $\gamma = \sqrt{\alpha^2 - \beta^2}$. Where the variance $\sigma^2$ for the exponential Lévy process model is as in the mean-variance mixture (see section 1.1). However it is possible to change $\gamma$ to any value without the structure of the distribution being altered and such that the price process is a martingale. Our choice of $\gamma$ makes sure that differences in pricing aren’t caused by distinguish stationary distribution of the variance $\sigma^2$, but by a time-dependency. Moreover it gives a similarity between the two measure transforms used.

Now by applying the above described change of measure and approximating the integral with a Riemann sum,

$$A(0) \approx e^{-rT} \mathbb{E} \left[ \max \left\{ \frac{S(0)}{N} \sum_{i=0}^{N} e^{X(t_i)} \Delta - K, 0 \right\} \right]$$

For simplicity we will work with a regular time partition $0 = t_0 < t_1 < \cdots < t_N = T$ with mesh $\Delta$. In the next section we will describe several methods do valuate the expectation in equation (7).

3. Simulation

There are two trends to simulate Inverse Gaussian random variates and processes. One is by exact simulation using general rejection method [24] and the other is a series expansion based on path rejection proposed by Rosinski [19]. Earlier Rosinski [19] had a technique using the inverse of the Lévy measure simulation using general rejection method [24] and the other is a series expansion based on path rejection.

3.1. Exponential Lévy process model. In the exponential Lévy process model $X(t)$ is simply an NIG random process. Hence we will describe a method to simulate a NIG Lévy process.

Since the NIG distribution is a mean-variance mixture (see Section 1.1) we can simulate a NIG random variate by,

- Sample $\sigma^2$ from $\text{IG}(\delta, \sqrt{\alpha^2 - \beta^2})$.
- Sample $\epsilon$ from $N(0, 1)$.
- $X = \mu + \beta \sigma^2 + \sigma \epsilon$.

By summing $\text{NIG}(\alpha, \beta, \Delta \delta, \mu)$ independent distributed increments one gets an NIG Lévy process on discrete points.

- Take $X(0) = 0$.
- For each increment, sample $x_i \sim \text{NIG}(\alpha, \beta, \Delta \delta, \mu)$.
- Set $X(t_i) = X(t_{i-1}) + x_i$.

To simulate $\sigma^2$ from an $\text{IG}(\delta, \gamma)$ distribution we will use a generator proposed by Michael, Schucany and Haas [14] (from now on referred to as MSH-method).

- Generate a random variate $Y$ with density $\chi^2$.
- Set $y_1 = \frac{\delta}{\gamma} + \frac{\gamma}{\gamma} - \frac{\gamma}{\gamma} \sqrt{4\delta\gamma Y + Y^2}$.
- Generate a uniform $[0, 1]$ random variate $U$ and if $U \leq \frac{\delta}{\delta + \gamma y_1}$, set $\sigma^2 = y_1$.
  If $U > \frac{\delta}{\delta + \gamma y_1}$, set $\sigma^2 = \frac{\delta^2}{\gamma y_1}$.

3.2. BNS SV model. By discretising time we may conclude that equation (4) can be rewritten into,

$$x(t) := \log S(t + \Delta) - \log S(t) \overset{D}{=} \mu \cdot \Delta + \beta \sigma^2(t) \cdot \Delta + \sqrt{\sigma^2(t)\Delta} \cdot \epsilon$$
where $\epsilon$ is a standard normal distributed random variable. This is based on the fact that for a Brownian motion $B$, $B(t + \Delta) - B(t)$ is equal in distribution to the random variate $\epsilon \sqrt{\Delta}$. We can use the following algorithm to generate log-returns $x$ on discrete points.

- Generate a sample path $\sigma^2_j(t_i)$ from IG-OU($\delta$, $\sqrt{\alpha^2 - \beta^2}$, $\lambda_j$) process, for $i = 0, \ldots, n$ and $j = 1, \ldots, m$.
- Set $\sigma^2_j(t_i) = \sum_{j=1}^{m} a_j \sigma^2_j(t_i)$ for $i = 0, \ldots, n$.
- Sample $\{\epsilon_i\}_{i=0}^{n}$ as a sequence i.i.d standard normal variables.
- Set $x_i = \mu \cdot \Delta + \beta \sigma^2(t_i) \cdot \Delta + \sigma(t_i) \cdot \epsilon \sqrt{\Delta}$, for $i = 0, \ldots, n$.

Again by summing the increments one gets the process $X$ on discrete points.

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**Figure 2.** Sample path of an IG-OU process.

We now focus on methods to simulate $\sigma^2$ from an IG-OU($\delta$, $\sqrt{\alpha^2 - \beta^2}$, $\lambda$) process. A solution to an SDE of the Ornstein-Uhlenbeck type,

$$d\sigma^2(t) = -\lambda \sigma^2(t)dt + d\eta(t)$$

is given by,

$$\sigma^2(t) = e^{-\lambda t} \sigma^2(0) + \int_0^t e^{-\lambda(t-s)}d\eta(s)$$

Moreover, up to indistinguishability, this solution is unique (See Sato [21] and Barndorff-Nielsen [3]). For an IG($\delta, \gamma, \lambda$)-OU process, $\sigma^2(t)$ has stationary marginal law IG($\delta, \gamma$). So $\sigma^2(0)$ is IG($\delta, \gamma$) distributed. Hence the most difficult term to simulate in (9) is the integral $\int_0^t e^{-\lambda(t-s)}d\eta(s)$. We will consider two methods to simulate from an IG-OU process.

3.2.1. Exact simulation. Take,

$$\sigma^2(\Delta) = \int_0^\Delta e^{-\lambda(\Delta-s)}d\eta(s)$$

As shown in Zhang & Zhang [24], for fixed $\Delta > 0$ the random variable $\sigma^2(\Delta)$ can be represented to be the sum of an inverse Gaussian random variable and a compound Poisson random variable in distribution, i.e.,

$$\sigma^2(\Delta) \overset{D}{=} W_0^\Delta + \sum_{i=1}^{N_\Delta} W_i^\Delta$$

where $W_0^\Delta \sim$ IG($\delta(1 - e^{-\frac{1}{2}\lambda\Delta}), \gamma$), random variable $N_\Delta$ has a Poisson distribution of intensity $\delta(1 - e^{-\frac{1}{2}\lambda\Delta}) \gamma$ and $W_1^\Delta, W_2^\Delta, \ldots$ are independent random variables having a common specified density function,

$$f_{W}(w) = \begin{cases} \frac{\gamma^{-1}}{\sqrt{2\pi}} w^{-3/2}(e^{\frac{1}{2}w\lambda\Delta} - 1)^{-1} \left( e^{-\frac{1}{2}\gamma^2w} - e^{-\frac{1}{2}\gamma^2we^{\lambda\Delta}} \right) & \text{for } w > 0, \\ 0 & \text{otherwise} \end{cases}$$
Moreover for any \( w > 0 \) the density function \( f_{W^\Delta}(w) \) satisfies,

\[
f_{W^\Delta}(w) \leq \frac{1}{2} \left( 1 + e^{\frac{1}{2} \lambda \Delta} \right) \left( \frac{\lambda}{\Gamma(\frac{\lambda}{2})} \right)^{1/2} w^{-(1 + \frac{1}{2})} e^{-\lambda \Delta w}.
\]

Hence we can use the rejection method on a \( \Gamma(\frac{1}{2}, 1/\gamma^2) \) distribution to simulate random variables \( W^\Delta \) with density function \( f_{W^\Delta}(w) \).

- Generate a \( \Gamma(\frac{1}{2}, 1/\gamma^2) \) random variate \( Y \).
- Generate a uniform \([0, 1]\) random variate \( U \).
- If \( U \leq \frac{f_{W^\Delta}(w)}{\int_0^\infty f_{W^\Delta}(w) \, dw} \), set \( W^\Delta = Y \), where \( g(Y) = \frac{\lambda}{\Gamma(\frac{\lambda}{2})} Y^{-(1 + \frac{1}{2})} e^{-\lambda \Delta Y} \).
- Otherwise return to the first step.

Since \( \sigma^2 \) is a stationary process we can conclude with equation (9), (10) and (11) that for all \( t > 0 \) we have the following equality in distribution,

\[
\sigma^2(t + \Delta) \overset{D}{=} e^{-\lambda \Delta} \sigma^2(t) + W^\Delta_0 + \sum_{j=1}^{N^\Delta_j} W^\Delta_j
\]

which can be translated in the following algorithm to generate a random variate \( \sigma^2(t_i) \) given the value of \( \sigma^2(t_{i-1}) \).

- Generate a \( \Gamma(\delta(1 - e^{-\frac{1}{2} \lambda \Delta}), \gamma) \) random variate \( W^\Delta_0 \).
- Generate a random variate \( N^\Delta \) from the Poisson distribution with intensity \( \delta(1 - e^{-\frac{1}{2} \lambda \Delta}) \gamma \).
- Generate \( W^\Delta_1, W^\Delta_2, \ldots, W^\Delta_{N^\Delta} \) from the density \( f_{W^\Delta}(w) \) as independent identically distributed random variates.
- Set \( \sigma^2(t_i) = e^{-\lambda \Delta} \sigma^2(t_{i-1}) + \sum_{j=0}^{N^\Delta_j} W^\Delta_j \).

Moreover \( \sigma^2 \) is a stationary process hence the initial value \( \sigma^2(0) \) can be generated from the density \( \Gamma(\delta, \gamma) \) using the MSH-method.

3.2.2. Series Representation. Alternatively Rosinski’s series representation (see [17]) can be used instead of the exact simulation. An \( \Gamma(\delta, \gamma) \) Lévy process can be approximated by,

\[
Y(t) = \sum_{i=1}^{\infty} \min \left\{ \frac{\delta T}{a_i}, e_i \tilde{u}_i^2 \right\} 1_{(\tilde{u}_i T < t)}
\]

where \( \{\tilde{u}_i\} \) is a sequence of independent exponential random numbers with mean \( \frac{\delta}{a_i} \), \( \{e_i\} \) and \( \{u_i\} \) are sequences of independent uniforms and \( a_1 < a_2 < \ldots < a_i < \ldots \) are arrival times of a Poisson process with intensity parameter \( 1 \). The series is converging uniformly from below. How large \( n \) should be depends on the parameters \( \delta \) and \( \gamma \). The summation should generally run over 1,000 to 100,000 terms.

From the definition of a Lévy process we may conclude that if we take \( T = 1 \) then \( Y(1) \) is an \( \Gamma(\delta, \gamma) \) random variable. Hence one can take \( \sigma^2(0) \overset{D}{=} Y(1) \) with \( T = 1 \).

In case of an \( \Gamma(\delta, \gamma, \lambda) \)-OU process the BDLP \( Z \) is the sum of two independent Lévy processes \( Z(t) = Z^{(1)}(t) + Z^{(2)}(t) \), where \( Z^{(2)}(t) \) is an \( \Gamma(\delta/2, \gamma) \) Lévy process, while \( Z^{(1)}(t) \) is a compound Poisson process of the form,

\[
Z^{(1)}(t) = \gamma^2 - \sum_{i=1}^{N(t)} v_i^2
\]

where \( N(t) \) is a Poisson process with intensity \( \frac{\gamma^2}{\lambda} \) and \( v_i \)'s are independent standard normal variables independent of \( N(t) \) (See Barndorff-Nielsen and Shephard [2]). Hence by using the series approximation (12) we can conclude that the following uniform convergence in \( t \in [0, T] \) \( (n \to \infty) \) holds (See Rosinski [19])

\[
e^{-\lambda T} \int_0^t e^{\lambda s} ds \left[ \sum_{i=1}^{\lfloor N(t) \rfloor} \frac{v_i^2}{\gamma} \right] + e^{-\lambda T} \int_0^t e^{\lambda s} ds \left[ \sum_{i=1}^n \min \left\{ \frac{1}{2}, \left( \frac{\lambda T}{a_i} \right)^2, e_i \tilde{u}_i^2 \right\} 1_{(u_i T < \lambda s)} \right] \overset{D}{=} \int_0^t e^{-\lambda(t-s)} dZ(\lambda s).
\]
Hence
\[
\int_0^t e^{-\lambda(t-s)} dZ(\lambda s) \overset{D}{=} e^{-\lambda t} \sum_{i=1}^{N(\lambda t)} \frac{v_i^2}{\gamma} e^{d_i} + e^{-\lambda t} \sum_{i=1}^{\infty} e^{\lambda \bar{u}_i d} \min \left\{ \frac{1}{2\pi} \left( \frac{\delta T}{a_i} \right)^2, e^{i \bar{u}_i^2} \right\} 1_{(u_i T < \tau)}
\]

All variables are as before. Moreover \{\bar{u}_i\} is a sequence of independent uniform random numbers and \(d_1 < d_2 < \cdots\) interarrival times of the Poisson process \(N(t)\).

By using equation (9) we can conclude that an IG-OU(\(\delta, \gamma, \lambda\)) process \(\sigma^2(t)\) can be approximated by,
\[
\sigma^2(t) \overset{D}{=} e^{-\lambda t} \sigma^2(0) + e^{-\lambda t} \sum_{i=1}^{N(\lambda t)} \frac{v_i^2}{\gamma} e^{d_i} + e^{-\lambda t} \sum_{i=1}^{\infty} e^{\lambda \bar{u}_i d} \min \left\{ \frac{1}{2\pi} \left( \frac{\delta T}{a_i} \right)^2, e^{i \bar{u}_i^2} \right\} 1_{(u_i T < \tau)}
\]

In contrast to the previous method the whole path is simulated directly.

4. Comparison of Simulation Methods

In Figure 3 we made quantile-quantile plots of the increments of the BNS SV model versus NIG random variates. One can see that exact simulation method and the series representation produce series from the same distribution. Moreover the increments of the BNS SV model are approximately NIG distributed. The parameters have a large influence how the returns look in comparison to NIG random variates, especially in the case of series representation. In the case of series representation the size of \(\delta\) has a large influence on the distribution of the returns. The sampling with series representation is based on the fact that an IG Lévy process can be approximated with (see (12)),
\[
Y(t) = \sum_{i=1}^{\infty} \min \left\{ \frac{2}{\pi} \left( \frac{\delta T}{a_i} \right)^2, e^{i \bar{u}_i^2} \right\} 1_{(u_i T < \tau)}.
\]

From the definition of a Lévy process we know that the increments should be inverse Gaussian distributed. But if we plot the empirical cumulative distribution function of the increments from a sample of 1,000 points, and compare it with the theoretic cumulative distribution function of the IG distribution, then there is a large difference (See Figure 4). This difference increases with the value of \(\delta\).

Financial assets have normally a value of \(\delta\) between 0.01 and 0.05. In this case the empirical distribution function and the theoretic distribution function coincide. However this still doesn’t make the simulation method reliable.

\[\text{Figure 3. Quantile-quantile plot of 10000 simulated points with parameters given by,} \]
\[\alpha = 64.0317, \beta = 4.3861, \delta = 0.0157, \lambda_1 = 0.0299, \lambda_2 = 0.3252, a_1 = 0.9122 \text{ and } a_2 = 0.0878.\]
(a) with $\delta = 0.05$ and $\gamma = 70$.

(b) with $\delta = 3$ and $\gamma = 5$.

**Figure 4.** Empirical cumulative distribution function of a 1000 points sample generated with series representation compared with the theoretic cumulative IG-distribution function.

The analysis resulting in Figure 4 is done with $n = 100,000$ to approximate the infinite sum in the series representation. As mentioned before it is hard to decide how large $n$ should be. It depends on the parameter values of $\delta$ and $\gamma$. Increasing the value of $n$ reduces the error. However the size of $n$ makes a huge difference in processing time using series representation as Table 1 shows.

The simulation times in Table 1 are measured in seconds on a not too new 1.3 Ghz i-book G4. The values are taken as the mean of 100 times simulating an IG-OU process.

One can see that in all cases the exact simulation is much faster. So not only in the precision, but also in simulation time the exact simulation method excels.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>$n$</th>
<th>Series representation</th>
<th>Exact simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1,000</td>
<td>0.0111</td>
<td>0.0025</td>
</tr>
<tr>
<td>20</td>
<td>10,000</td>
<td>0.1382</td>
<td>0.0028</td>
</tr>
<tr>
<td>20</td>
<td>100,000</td>
<td>1.2595</td>
<td>0.0026</td>
</tr>
<tr>
<td>40</td>
<td>1,000</td>
<td>0.0185</td>
<td>0.0044</td>
</tr>
<tr>
<td>40</td>
<td>10,000</td>
<td>0.2709</td>
<td>0.0044</td>
</tr>
<tr>
<td>40</td>
<td>100,000</td>
<td>2.7410</td>
<td>0.0053</td>
</tr>
<tr>
<td>60</td>
<td>1,000</td>
<td>0.0288</td>
<td>0.0059</td>
</tr>
<tr>
<td>60</td>
<td>10,000</td>
<td>0.3429</td>
<td>0.0051</td>
</tr>
<tr>
<td>60</td>
<td>100,000</td>
<td>4.3102</td>
<td>0.0060</td>
</tr>
</tbody>
</table>

5. **Pricing Asian options**

We consider the problem of pricing Asian options written on an asset dynamics given by an exponential NIG-Lévy process resp. the BNS SV model. We will handle a representative case of pricing with calibrated parameters on log-returns of the AEX-index. For simplicity we assume that the stock price today is $S(0) = 100$ and that the risk-free interest rate is $r = 3.75\%$ yearly.

After calibration on a set of daily return data of the AEX-index the parameters are given by,

$$\hat{\alpha} = 94.1797 \quad \hat{\beta} = -16.0141 \quad \hat{\delta} = 0.0086 \quad \hat{\mu} = 0.0017$$

So under the risk neutral measure $Q$ we have,

$$\alpha = 94.1797 \quad \beta = -17.3709 \quad \delta = 0.0086 \quad \mu = 0.0017$$
in the exponential Lévy process case and,

$$\beta = -\frac{1}{2}, \quad \delta = 0.0086, \quad \mu = r = \log(1.0375^{1/365}), \quad \gamma = \sqrt{\alpha^2 - \beta^2}.$$

in the BNS SV case. Moreover by least square fitting with 1 superposition $\lambda$ is given by,

$$\lambda = 0.0397$$

With the above parameters we priced Asian options with a common strike $K = 100$ and exercise horizons of four, eight, or twelve weeks (See Table 2).

<table>
<thead>
<tr>
<th>$T$</th>
<th>Lévy Pr. Mean (conf. interval)</th>
<th>Series repr. Mean (conf. interval)</th>
<th>Exact Mean (conf. interval)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.0539 (1.0441, 1.0633)</td>
<td>0.9878 (0.9777, 0.9977)</td>
<td>0.9843 (0.9743, 0.9943)</td>
</tr>
<tr>
<td>20</td>
<td>1.0581 (1.0477, 1.0676)</td>
<td>0.9908 (0.9813, 1.0012)</td>
<td>0.9809 (0.9702, 0.9909)</td>
</tr>
<tr>
<td>20</td>
<td>1.0542 (1.0442, 1.0641)</td>
<td>0.9879 (0.9770, 0.9975)</td>
<td>0.9754 (0.9655, 0.9853)</td>
</tr>
<tr>
<td>40</td>
<td>1.5097 (1.4966, 1.5231)</td>
<td>1.4054 (1.3916, 1.4195)</td>
<td>1.4209 (1.4064, 1.4345)</td>
</tr>
<tr>
<td>40</td>
<td>1.5109 (1.4977, 1.5248)</td>
<td>1.4027 (1.3888, 1.4168)</td>
<td>1.4023 (1.3893, 1.4168)</td>
</tr>
<tr>
<td>40</td>
<td>1.5141 (1.5004, 1.5286)</td>
<td>1.3939 (1.3807, 1.4079)</td>
<td>1.4133 (1.3994, 1.4281)</td>
</tr>
<tr>
<td>60</td>
<td>1.8684 (1.8519, 1.8850)</td>
<td>1.7518 (1.7343, 1.7682)</td>
<td>1.7566 (1.7382, 1.7742)</td>
</tr>
<tr>
<td>60</td>
<td>1.8857 (1.8693, 1.9049)</td>
<td>1.7659 (1.7481, 1.7822)</td>
<td>1.7525 (1.7349, 1.7698)</td>
</tr>
<tr>
<td>60</td>
<td>1.8751 (1.8576, 1.8913)</td>
<td>1.7346 (1.7169, 1.7520)</td>
<td>1.7439 (1.7263, 1.7617)</td>
</tr>
</tbody>
</table>

The mean of the option price is taken over 100,000 simulations. The confidence intervals are calculated by bootstrap using 2,000 resamples. Moreover to approximate the infinite sum in the series representation, $n$ is taken to be 1,000.

The exponential Lévy process model is overpricing in comparison to the BNS SV model. The empirical distribution of the generated option prices is similar for the two simulation methods of the BNS SV model. A quantile-quantile plot shows the last property clearly (see Figure 5). A Kolmogorov-Smirnov test confirms that the generated prices from the two simulation methods come from the same continuous distribution at a 5% significance level. The same test rejects the hypothesis that the generated prices of the exponential Lévy process model and the generated prices of the BNS SV model come from the same continuous distribution.

On the interval [0, 6] consists in all cases of appro the exponential Lévy price is slightly higher then the BNS SV price. Moreover the interval [0, 6] consists of approximately 95% of the simulated points, hence the difference in prices on this interval leads to a slight higher mean in the exponential Lévy process model. In the other 5% of the cases the BNS SV model prices higher and sometimes even extensively higher.

When alternating with the strike price $K$, the relation between the option prices of the models is changing (see Figure 6(a), 6(b)). The corresponding means of the option prices for 100,000 times simulating are given by,

<table>
<thead>
<tr>
<th>$K$</th>
<th>Lévy Pr. Mean (conf. interval)</th>
<th>Series repr. Mean (conf. interval)</th>
<th>Exact Mean (conf. interval)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1.5048 (1.4917, 1.5191)</td>
<td>1.4074 (1.3925, 1.4222)</td>
<td>1.4029 (1.3885, 1.418)</td>
</tr>
<tr>
<td>110</td>
<td>0.0077 (0.0068, 0.0087)</td>
<td>0.0294 (0.0270, 0.0325)</td>
<td>0.0294 (0.0271, 0.0320)</td>
</tr>
</tbody>
</table>
In case of a strike $K = 80$ the mean option prices are approximately the same, but the distribution of the price processes from the different models seem totally not to coincide (see Figure 6(a)). However approximately 90% of the simulated prices lie in the vertical part between 15 and 26. Hence the difference in pricing of the two models is caused by the outliers. The BNS SV model exaggerates the price of the extremes in comparison to the exponential Lévy process model i.e. BNS SV model will price low priced outliers even lower and high priced outliers higher compared to the exponential Lévy process model.

At least 95% of the of the simulated prices is zero in the case of a strike $K = 110$. So in the QQ-plot (Figure 6(b)) it is again visible that the BNS SV model exaggerates the extreme values of the pricing process compared to the exponential Lévy process model.

One can speculate that the difference in the outliers is caused by volatility. A period with low return variance can lead to a more extreme price of the option.
models stochastic volatility.

Although the algorithm of an IG-OU process based on series representation of Rosinski [19] is popular, it is questionable whether it is reliable. The algorithm is based on a method to simulate an inverse Gaussian Lévy process from which the increments not always follow an IG law. To reduce this error the number of iterations should be increased. This has a large influence on the simulation time. Hence the exact simulation of Zhang & Zhang [24] is faster and more accurate. The log-returns of the BNS SV model are approximately NIG distributed. How they approximate an NIG distribution depends on the parameters.

The price processes of the two simulation methods behave similarly. In pricing the exponential Lévy process model slightly overprices in comparison to the BNS SV model. The BNS SV price process exaggerates the extremes. Meaning that the BNS SV model will price low priced outliers even lower and high priced outliers higher compared to the exponential Lévy process model.

One can speculate that the difference in pricing of the outliers is caused by volatility. A period with low return variance can lead to a more extreme price of the option.

REFERENCES
