Solving a Deconvolution Problem in Photon Spectrometry


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1. Introduction

It is a well known fact that the energy spectra measured in the laboratory always differ to a larger, or smaller extent, from the

“true” spectra—that are the object of the ultimate physical interest—and in order to infer the latter the results of the
measurements must be corrected by including the finite resolution of the measuring devices. A very similar situation occurs
when one wants to extract the spectrum of the neutral pions by observing the spectrum of the gammas resulting from the pion

abstract

We solve numerically a deconvolution problem to extract the undisturbed spectrum from the measured distribution contaminated by the finite resolution of the measuring device. A problem of this kind emerges when one wants to infer the momentum distribution of the neutral pions by detecting the p0 decay photons using the photon spectrometer of the ALICE LHC experiment at CERN [1]. The underlying integral equation connecting the sought for pion spectrum and the measured gamma spectrum has been discretized and subsequently reduced to a system of linear algebraic equations. The latter system, however, is known to be ill-posed and must be regularized to obtain a stable solution. This task has been accomplished here by means of the Tikhonov regularization scheme combined with the L-curve method. The resulting pion spectrum is in an excellent quantitative agreement with the pion spectrum obtained from a Monte Carlo simulation.

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decay and measured with the Photon. Spectrometer in the ALICE LHC experiment at CERN [1].

PHOton Spectrometer (PHOS) is the dedicated photon detector of ALICE experiment which has high spatial and energetic resolution. It is situated at the bottom of the ALICE detector barrel at a distance of 4.6 m from the interaction point and covers 100° of azimuthal angle and 0.24 units of rapidity in the mid rapidity region. PHOS is designed for detection of high-energy photons and can as well be used for identification of short-lived particles which decay into photons. Neutral pions, which have very short lifetime and decay prior to interacting with any detector are identified in PHOS by decay photon pairs that have pair invariant mass equal to the value of the mass of neutral pion. The total energy-momentum of a photon is found by adding energies and momenta of decay photons. Not all neutral pions are registered in the PHOS. In addition to other decay channels which can not be detected by the photon detector a very important factor is its limited geometrical and physical acceptance. Some π⁰ are "lost" since one of the decay photons may fly past the solid angle covered by the detector aperture or undergo conversion into an electron–positron pair before reaching the detector. Another factor influencing the measured particle spectra is the "blurring" in the detector itself which is analogous to the chromatic aberration in optics when a mono-energetic wave after going through the detector is split into a set of waves with different energies. The measured energy of a π⁰ is different from its true value leading to the distortion of its energy distribution which introduces an error in measuring the particle spectra. Our task is to study the possibility of extracting the true spectrum of neutral pions from the measured one and the purpose of this work is to find a practical solution of this problem.

In mathematical terms, the measured distribution is given as a convolution [2] of the undistorted distribution with a known response function accounting for the finite resolution of the measuring device. The objective is to get the undistorted distribution and the latter can be inferred by inverting the integral transform, which is equivalent to solving the appropriate Fredholm integral equation of the first kind. Such procedure, sometimes called deconvolution, belongs to the general category of inverse problems encountered in all modern developments of the natural sciences (cf. [3] and references therein). The applied inverse problems typically involve the indirect estimation of certain quantities characterizing a given object or medium from the observation of a response of this object or medium to a probing signal. Big demand from applications stimulated the development of numerical techniques and for a couple of decades the field of inverse problems has been the fastest growing branch of applied mathematics (a very lucid introduction to this subject can be found in [4]). Perhaps not surprisingly, in most cases deconvolution problems cannot be solved analytically, and therefore, one has to resort to numerical methods. Upon introducing a numerical quadrature, the underlying integral equation is discretized and the problem is thereby reduced to that of solving a linear system of algebraic equations. Unfortunately, this algebraic problem is ill-posed since the resulting coefficient matrix is of ill-determined rank and such matrices are known to be severely ill-conditioned and may be singular. These difficulties stem from the fact that the deconvolution problem itself is ill-posed. The very concept of a well-posed problem is quite old and goes back to the Hadamard [5] formulation where the problem is said to be well-posed if: (i) the solution exists, and (ii) the solution is unique, and (iii) the solution depends upon the input data in a continuous way. The problem is called ill-posed when one (or more) of these conditions are violated. In practice, the first two conditions can be often stipulated by enlarging, or reducing the admissible solution space, but the enforcement of the stability condition (iii) turns out to be the most troublesome. Indeed, when the condition (iii) does not hold a small perturbation in the data might be amplified generating large perturbations in the solution. Stability could be restored only by changing the topology of the solution space and in most cases this is not possible owing to the presence of the experimental errors. In algebraic terms, the system matrix becomes under-determined due to the clustering of small singular values. Hence, the algebraic problem cannot be solved without an injection of an additional information about the solution. Usually, it is natural to expect from the solution to be smooth and incorporating such requirement helps to stabilize the problem. This procedure, known as regularization, has been used in this paper to solve the deconvolution problem for the ALICE photon spectrometer. We start by providing the mathematical background of the regularization method. Next, we check its performance by solving numerically a particular deconvolution problem where the solution is known analytically and reassured by the rather satisfactory outcome of this check, we apply with reasonable confidence the same method to solve the PHOS deconvolution problem. Finally, we make our last check confronting the deconvoluted PHOS spectrum with the Monte Carlo simulation.

Our paper is structured as follows. In Section 2 we outline the regularization procedure which is subsequently applied to a spectroscopic example of a line width problem where the Lorentz profile is convoluted with a Gaussian accounting for the Doppler broadening effect (cf. Section 3). Section 4 deals with the inverse problem for the ALICE photon spectrometer. A short summary of the whole work is given in Section 5.

2. Computational framework

In many types of experiments, notably in scattering and spectroscopic measurements of all kinds, a physical quantity, or a distribution measured in the laboratory does not provide yet the quantity of physical interest because these data are often non-negligibly distorted by the finite resolution of the measuring system. In the majority of these cases the distorted data can be expressed mathematically as a convolution of the undistorted function—that would have been measured had the measuring instrument had an infinite resolution—with the response, or resolution, function of the instrument. Since in practice increasing the resolution of an instrument is always limited, the uncontaminated data must be derived by inverting numerically the integral transform associated with the convolution. This procedure is called deconvolution and consists in solving the appropriate integral equation.

In mathematics, convolution is an operation on two functions $L(x)$ and $G(x)$ producing a third function $V(x)$ that is usually viewed as a weighted version of one of the original functions, whereas the remaining function is then responsible for the weighting. In this paper we consider exclusively real functions. The relation between these three functions takes a generic form:

$$V(x) = \int_{-\infty}^{\infty} L(x-y)G(y) \, dy = \int_{-\infty}^{\infty} G(x-y)L(y) \, dy$$

where the property expressed by the last equality is known as commutativity. A typical problem of this kind occurs in optical, or X-ray spectroscopy where $V(x)$ (known as the Voigt profile [6]) originates from the Lorentzian line shape $L(x)$ when the latter is viewed through a measuring instrument whose response $G(x)$ is of the Gaussian form. The independent variable $x$ denotes then the appropriate wave number. The observed line has the shape given as the convolution which is a particular kind of the integral
transform (1) and we shall elaborate on this interesting example later on.

In the deconvolution problem the functions \( V(x) \) and \( G(x) \) are regarded as known and the objective is to determine \( L(x) \) by inverting the integral transform (1). Although caution must prevail, it is instructive to try solving (1) directly by using the Nyström method [7]. To this end, we need a quadrature rule approximating the integral of a real function \( f(x) \) in the range \((a,b)\) by the generic expression of the form:

\[
\int_a^b f(x) \, dx \approx \sum_{i=1}^N w_i f(x_i)
\]

where the real numbers \( x_i \) and \( w_i \) are, respectively, the abscissas and the weighting factors whose values are provided in the tables for any quadrature order \( N \). These numbers can be also generated numerically [7] so that in the following they are regarded as known. We note that the popular quadrature rules such as rectangular, midpoint, trapezoidal, Simpson, Gaussian, etc. can all be cast to the form (2). Introducing a cut-off, whose size depends upon the nature of the problem, the improper integral occurring in (1) can be discretized with the aid of (2) and we end up with a linear system of \( N \) algebraic equations

\[
A \cdot \mathbf{w} - z = u
\]

where the boldfaced symbols denote \( N \times N \) matrices, \( A_i = G(x_i - x) \) and \( \mathbf{w} \) is a diagonal matrix containing the weights \( w_i \). The symbols \( u \) and \( z \), abbreviate \( N \)-dimensional vectors, containing the functions \( V(x) \) and \( L(x) \), respectively, evaluated at the mesh points \( x_i \). It can be easily verified that the problem of solving (3) is equivalent to finding the vector \( z \) that minimizes the residual norm functional

\[
\|f\|^2 = \int_a^b f(x)^2 \, dx = f^T \cdot \mathbf{W} \cdot f.
\]

Indeed, expanding (4) and equating to zero the derivative with respect to \( x^T \), we end up with the normal equations that are equivalent to (3). Unfortunately, when the order of discretization \( N \) increases the problem rapidly becomes ill-conditioned. As a result small changes in \( u \) are capable of inducing big changes in the solution \( z \) and a small experimental error, unavoidably present in the input, may push \( z \) far away from the true physical solution. There is a vast literature devoted to resolving this difficulty but perhaps the most popular methods are based on the ideas proposed by Phillips and Tikhonov [8] widely known today as Tikhonov regularization. The main idea behind this method consists in replacing an ill-posed problem of solving the Fredholm integral equation of the first kind, by a stable, closely related, minimization problem involving a small positive parameter \( \alpha \). To achieve this goal the functional (4) is extended by adding \( ad \) hoc an extra term which yields the following weighted combination of the residual norm and a side constraint

\[
\|f\|^2 + \alpha \cdot \|z\|^2 = \|A \cdot \mathbf{w} - z - u\|^2 + \alpha (z^T \cdot \mathbf{W} \cdot (z - z_0)^2)
\]

where \( \mathbf{W} \) is some, as yet unspecified, regularizing matrix (or Tikhonov operator). The vector \( z_0 \) represents an initial estimate of the true solution \( z \) but if the former is not available we may put \( z_0 = u \). Sometimes there are good reasons to believe that the convoluted function should be not much different from the undistorted function. In this case a reasonable choice would be \( z_0 = u \). The Tikhonov operator may be adopted in various forms depending on the nature of the problem. In most applications it is sufficient setting \( T \) to be equal to the unit matrix, with a possible scaling coefficient being absorbed into the constant \( \alpha \). The extra term in (5), added by hand, is essentially a measure of the “size” of the regularized solution but by including explicitly this term we deliberately favor only smooth solutions in which oscillations would be suppressed. It can be seen that, quite generally, the side constraint acts as a filter allowing to pick up a particular class of solutions.

Varying (5) with respect to \( \alpha \) we are led to a set of well-conditioned normal equations

\[
(A^T \cdot \mathbf{W} \cdot A + \alpha T^T \cdot \mathbf{W} \cdot T) \cdot \mathbf{w} - z = A^T \cdot \mathbf{w} - z + \alpha T^T \cdot \mathbf{W} \cdot T \cdot \mathbf{z} - z_0,
\]

from which the minimizing vector this time can be determined by standard methods of linear algebra.

It is quite obvious that the proper choice of the regularization parameter \( \alpha \) is an essential control the weight given to minimization of the side constraint relative to the minimization of the residual norm. Thus, in the extreme situation of a very large \( \alpha \), although a stable solution is guaranteed but this might be achieved at the expense of a large residual norm and the resulting \( z \) might differ appreciably from the exact solution. Going to the other extreme of a very small \( \alpha \) has the opposite effect because owing to ineffective regularization we might be faced with the situation where ill-conditioning of the system causes severe instabilities and discontinuous dependence of \( z \) on the input vector \( u \). Clearly, some intermediate value of \( \alpha \) is desired that would ensure a compromise between these two opposite tendencies. Several methods have been proposed seeking for such optimal value of the regularization parameter in a systematic way but perhaps the most convenient is a graphical tool known as the L-curve method. Formally, this method introduces two functions of the parameter \( \alpha \), denoted here as \( X(\alpha) \) and \( Y(\alpha) \) and given by the following expressions

\[
X(\alpha) = \|A \cdot \mathbf{w} - z_0 - u\|^2 = (A \cdot \mathbf{w} - z_0 - u)^T \cdot \mathbf{W} \cdot (A \cdot \mathbf{w} - z_0 - u)
\]

and

\[
Y(\alpha) = \|T \cdot \mathbf{w} - (z_0 - z_0)\|^2 = (T \cdot \mathbf{w} - (z_0 - z_0))^T \cdot \mathbf{W} \cdot (T \cdot \mathbf{w} - (z_0 - z_0))
\]

where \( z_0 \) is the solution of (6) for a fixed \( \alpha \). Since the functional (5) is a sum of two positive terms it is of interest to know the values of each term separately and the functions \( X(\alpha) \) and \( Y(\alpha) \) provide this information. When the parameter \( \alpha \) is allowed to vary, these two functions define parametrically a curve in the \((X,Y)\) plane and when plotted in the double log scale this curve usually has a characteristic “L” shape with a distinct kink separating the vertical and the horizontal segments. For very small \( \alpha \) the function \( Y(\alpha) \) shows a rapid variation whereas \( X(\alpha) \) shows little sensitivity to \( \alpha \) and this behavior is reflected by the vertical part of the curve. When \( \alpha \) becomes large enough this tendency is reversed since the solution has been stabilized and further increase of \( \alpha \) pushes \( z_0 \) away from the exact solution. This causes a rapid increase of the residual norm and is reflected by the horizontal segment of the L-curve. The sought for value of \( \alpha \) ensuring the best trade off between smoothness and fidelity is associated with the kink on the L-curve and may be determined from the plot in a systematic way.

3. Exactly solvable example

In order to get some sort of a feeling of how does the regularization scheme work in practice we are going to consider a particular example in which the solution of the deconvolution problem is known analytically. This gives us a unique opportunity where the numerically generated solution might be confronted
with the exact solution. Moreover, the deconvolution example to be considered is far from being academic as it is the line shape problem alluded to in the preceding section where the three functions \( L(x), G(x) \) and \( V(x) \) occurring in (1) are now assumed to be Lorentzian, Gaussian and Voigtian profiles, respectively. We set the Lorentzian distribution as

\[
L(x) = \frac{A_L}{(\Gamma/2)^2 + (x-x_0)^2}; \quad A_L = \frac{\Gamma}{2\pi}
\]

the Gaussian function

\[
G(x) = A_G e^{-x^2/(2\sigma^2)}; \quad A_G = \frac{1}{\sqrt{2\pi}\sigma}
\]

and the Voigt profile

\[
V(x) = \frac{A_V}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-(\nu-t)/(2\sigma^2)}}{\sqrt{\pi}} dt
\]

where

\[
a = \frac{1}{\sigma\sqrt{2}}, \quad \nu = \frac{x-x_0}{\sigma\sqrt{2}}.
\]

It has to be explained that although \( V(x) \) is given here in an integral form, this integral can be very accurately expressed in terms of known functions and the Voigt function can be efficiently generated numerically (cf. [6] and references therein).

Our numerical problem can be formulated as follows: for a given \( V(x) \) and \( G(x) \), solve the integral Eq. (1) to obtain \( L(x) \). It will be convenient for us, to adopt units in which \( \Gamma/2 = 1 \). In these units we take \( x_0 = 10 \) setting the Gaussian width to be \( \sigma\sqrt{2} = 1 \). With such integrand the improper integral occurring in (1) can be truncated and we took the integration limits to be \((0, 20)\) in our units and the problem can be discretized by introducing a numerical quadrature. We used three different quadrature rules: trapezoidal, midpoint and the Gaussian quadrature. The popular Simpson rule was also tried but the results were discouraging and eventually this quadrature had to be abandoned. With all quadratures the resulting matrix \( A \) is non-singular, but becomes ill-conditioned when the quadrature order \( N \) increases. For a modest value \( N = 50 \) the condition number of \( A \), i.e. the ratio of the biggest and the smallest singular values, is \( 10^{13} \) and the solution of the system (3) is unstable to small changes of the right-hand side. Therefore, it is impossible to obtain a stable solution without regularization. This situation is illustrated in Fig. 1 and one can see that despite of the fact that the system matrix is regular and invertible, a straightforward inversion yields nonsensical results.

In the implementation of the Tikhonov regularization one has to select the quadrature rule and set the value of the approximation order \( N \). Apart from that there is also the question what should be taken for the initial estimate of the solution \( z_0 \). If \( N \) is too large the solutions usually show some annoying oscillations which are difficult to get rid of. On the other hand, taking \( N \) too small, might result in inadequate precision and the desired accuracy will never be achieved. In order to shed some light on these questions we have examined the average relative error as a function of \( N \)

\[
\text{Error} = \frac{1}{N} \sum_{i=1}^{N} \left| 1 - \frac{L_{\text{reg}}(x_i, z)}{L(x_i)} \right|
\]

where \( L(x_i) \) and \( L_{\text{reg}}(x_i, z) \) are the exact and the retrieved solutions, respectively, evaluated at the mesh points \( x_i \). The solution generated numerically depends also upon the value of the regularization constant and in computing (13) always the optimal value of \( z \) was used. In Fig. 2 we plot the error (13) versus \( N \) for three different quadrature rules. It is apparent from this plot that, at least for the spectroscopic example under consideration, the midpoint rule turns out to be the winner. It is remarkable that the largest contribution to the relative error comes from a handful of entries responsible for the oscillations appearing at the end points. The amplitudes of these oscillations increase with \( N \) but going down with \( N \) requires some care if we do not want to upset the residual norm value. In Fig. 3 a plot is shown on the plane of the averaged \( J^2 \) function against the residual norm \( A \cdot W \cdot z - \text{vec}(u) \). The curve is parameterized with different values of the quadrature order \( N \). It may be inferred from this plot that the best choice is \( N = 75 \) as this corresponds to the point of curve which is nearest to the origin. To find the optimal value of the regularization parameter we use the method of L-curve [9]. To produce the L-curve the expressions (7) and (8) must be computed perpetually for different values of the parameter \( z \) and the actual L-curve is displayed in Fig. 4. The values of \( z \) have been taken from the interval between \( 10^{-12} \) and 10 and are specified on the plot. Actually, in the problem under consideration the kink has been rounded and the curve only remotely reminds the letter L. Nevertheless, all the qualitative features are apparent from this plot. Taking \( z \) to be extremely small there is almost no filtering and although our solution is quite accurate (small

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**Fig. 1.** Naive deconvolution attempt without regularization. The exact solution (bell-shaped curve) has been magnified by a factor of 20 to match the scale.

**Fig. 2.** Average relative error versus quadrature order \( N \) for three different integration rules.

**Fig. 3.** A plot is shown on the plane of the averaged \( J^2 \) function against the residual norm \( A \cdot W \cdot z - \text{vec}(u) \). The curve is parameterized with different values of the quadrature order \( N \).

**Fig. 4.** The actual L-curve is displayed, with the values of \( z \) specified on the plot.
residual norm) but $z_a$ is badly contaminated with rapidly oscillating components which is manifested by a sizable value of the norm. The remedy is to increase $z$ which acts as a filter suppressing the oscillating components and leaving only the smooth part of the solution. As a result, the norm decreases rapidly, whereas the residual norm stays almost unchanged. This behavior persists in a wide range of values until $z$ reaches some critical point at which this tendency is reversed. Further increase of $z$ brings no improvement in suppressing the norm value and at the same time the residual norm rapidly increases. Clearly, too much filtering eventually affects also the smooth component deteriorating the accuracy of the solution (large residual norm).

The optimal value of the regularization parameter in our example is $z = 10^{-6}$ and this value has been derived by locating the maximum curvature point of the graph.

In Fig. 3 we compare the exact and the numerical solution corresponding to the optimal value of the regularization parameter for the midpoint quadrature rule. The other quadratures gave very similar results and there is not much point to present them here. Our simple example shows that the regularization method works quite well and can be applied with confidence to the PHOS problem.

4. The inverse problem for the photon spectrometer

In this section the regularization scheme will be applied to deduce the energy distribution of the neutral pions using as input the photon spectrum measured with the PHOS detector. Our aim is to retrieve the undisturbed energy distribution of $\pi^0$'s.

The relation between the true and the measured $\pi^0$ energy spectra for the PHOS problem is similar to (1) and reads

$$f_{\text{meas}}(x) = \int_{-\infty}^{+\infty} G(x-y)p(y)f_{\text{true}}(y)\,dy$$

(14)

where $f_{\text{true}}(x)$ and $f_{\text{meas}}(x)$ are the true and the measured spectra, respectively. The function $G(x-x')$ describes the blurring of the particle energy in the detector and $p(x)$ accounts for the effects of the energy loss in the medium and the imperfect detection of particles due to the limited aperture of the detector. While the function $G(x-x')$ is symmetric and normalized to unity the presence of $p(x)$ destroys the symmetry and changes the normalization of the total kernel in the integral Eq. (14). Nevertheless, (14) can be brought to the standard convolution form by introducing a new unknown function $g(x) = p(x)f_{\text{true}}(x)$. Therefore, we can use the formalism outlined in Section 2.

In order to find the matrix elements of $A$, being a discrete counterpart of the kernel $G$ occurring in (6), we used the Aliroot software package, which is the ALICE offline framework for simulation, reconstruction and analysis based on the ROOT system incorporating programming packages for event generation and simulation of a particle transport in the medium [10]. The simulation is done by generating the primary and the final-state particles taking into consideration the detector geometry and the properties of the detector medium. The photons hitting the detector produce an output signals which are processed during the reconstruction phase to identify the neutral pions and to find their energies.

To accomplish our task a sample of $\pi^0$'s has been produced by the PYTHIA event generator and subsequently filtered through GEANT3-based PHOS filter. The latter generates the spectrometer response taking into account the geometrical configuration of the PHOS modules, material properties of its sensitive elements and those of the surrounding medium consisted of the air and the bulk parts of all detectors. As an input to the PHOS filter we used two samples $3 \times 10^6$ and $10^6 \pi^0$'s generated with the uniform and an exponential energy distribution spanning the $0$–$30$ GeV interval with uniformly distributed azimuthal and polar angles. The final
state photons descending from $\pi^0$ decay reaching the detector were reconstructed and identified by clusters of adjacent elementary cells of the PHOS excited by the incoming photons, the total magnitude of the signal from the cluster giving the energy of the photon. In subsequent analysis the neutral pions were identified by the invariant masses of the photon pairs. The sum of energies of such a pair of photons gives the total energy of the identified $\pi^0$.

Fig. 6 shows the input energy distribution of the generated $\pi^0$-s together with that for $\pi^0$-s filtered through the PHOS. In order to retrieve the input distribution using the filtered one we need to solve the integral Eq. (14). As in the case of the Voigt deconvolution we begin with the discretization of the entire energy interval by introducing a number of small subintervals. The energy distributions of the input and the filtered $\pi^0$-s are used to determine the matrix elements of $A$. The actual size of the elementary subintervals can be adjusted depending on the nature of the problem. For instance, in the case of low-energy particles with rich statistics we shall need a fine mesh with bin size of the same order as the energy resolution of the detector. By contrast, for high energies with only a few particles emerging per unit interval, the bin size can be taken substantially larger. To give an illustration of how the PHOS device matrix $A$ distorts true spectrum we present in Fig. 7 the energy distribution resulting from filtering through the PHOS detector a bunch of mono-energetic input $\pi^0$-s with energy of 21 GeV. The filtered spectrum is asymmetric with a half-height width of about 0.2 GeV. This spectrum can be viewed as the result of action of the $A$-operator on the input data vector with a single non-zero component. Clearly, our problem is similar to that considered in Section 3 except for the fact that now the shape of the energy profile describing the detector response varies with energy and is not Gaussian. The function $p(x)$ introduced above, present in the definition of $g(x)$, accounts for the energy dependence of the PHOS efficiency. From Fig. 8 one sees that the efficiency is small for low energies and increases with the energy, reaching a broad peak at 15–25 GeV. This shape can be explained by the fact that the photons originating from the decaying low-energy pions have a too wide opening angle preventing them to be both registered, in view of the limited aperture of the detector. The efficiency is dropping down after going through the maximum at 15–25 GeV. This phenomenon has an alternative geometrical explanation: the opening angle of the decay photons from the high momentum...
The scheme of finding the regularized solution for the PHOS problem is similar to that outlined in the preceding section. For illustrative purposes we present an example with the quadrature order \( N = 300 \) which ensures the desired accuracy. Therefore, the resulting linear system is of the size \( 300 \times 300 \). Before applying the regularization scheme it is instructive trying a direct inversion of the matrix \( A \) and the results are presented in Figs. 9 and 10.

Owing to the strong oscillations present in the retrieved spectra the latter have a totally different shape from the appropriate input distribution with the respective amplitudes differing by up to four orders of magnitude. Thus, the need of some regularization is imperative.

Encouraged by the success of the deconvolution of the Voigt spectrum, we apply again the Tikhonov regularization introducing the regularization parameter \( \alpha \) (5) and repeating all the steps from the preceding section. From the appropriate L-curve analysis (cf. Figs. 11 and 12) we obtained \( \alpha = 10^{-8} \) and \( \alpha = 10^{-12} \) for the uniform and the exponential distribution, respectively. In the particular case under consideration the right choice of \( \alpha \) is rather obvious as the kink of the L-curve is very pronounced.
Admittedly, the L-curves presented in Figs. 11 and 12 depart from a typical L-shape, but nevertheless they have all the necessary features expected from the L-curve and discussed in Section 2. The results of our computations are presented in Figs. 13 and 14 which show both, the input distributions, and the retrieved spectra, respectively. It is apparent from Figs. 13 and 14 that the oscillating components present in Figs. 9 and 10 have been suppressed. At the same time, the residual norm of the ultimate solution has been kept at a sufficiently low level ensuring that the problem at hand has been solved with sufficient accuracy. The results displayed in Figs. 13 and 14 demonstrate that the regularization technique works very well indeed. The retrieved $\pi^0$ energy spectra are in an excellent agreement with the input $\pi^0$ energy distribution.

5. Summary

The purpose of this work was to provide a practical calculational scheme for deriving the $\pi^0$ momentum distribution from the measured $\pi^0$ decay photon spectrum. Formally, the sought for $\pi^0$ spectrum can be obtained from the measured photon spectrum by inverting the integral transform involving the response function accounting for the finite resolution and efficiency of the measuring device which is equivalent to solving the Fredholm integral equation of the first kind. The above inverse problem is known to be ill-posed and in consequence a direct numerical solution would be highly unstable. Therefore, in order to obtain a physically meaningful solution we apply Tikhonov regularization combined with the L-curve method. The effectiveness of the applied procedure has been first verified on an exactly soluble example and then the method was used to get the $\pi^0$ distribution from the simulated measurements of the photon spectrum in the PHOS detector of the ALICE LHC CERN experiment. As a final test, the deduced $\pi^0$ distribution was confronted with the result from the Monte Carlo simulated input.

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