Note

Reachability in live and safe free-choice Petri nets is NP-complete

Javier Esparza

Institut für Informatik, Technische Universität München, Arcisstr. 21, D-80290 München, Germany

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Abstract

The complexity of the reachability problem for live and safe free-choice Petri nets has been open for several years. Several partial results seemed to indicate that the problem is polynomial. We show that this is unlikely: the problem is NP-complete. © 1998—Elsevier Science B.V. All rights reserved.

1. Introduction

Free-choice Petri nets were first defined and studied in the early seventies [1, 6]. Today, they are accepted as the largest class of Petri nets for which relevant analysis problems can be solved in polynomial time. A series of papers, starting with [4] and culminating with [8], has shown that the problem of deciding if a free-choice Petri net is live and bounded can be solved in $O(nm)$ time, where $n$ and $m$ are the number of places and transitions of the net, respectively. In turn, many analysis problems of live and bounded free-choice Petri nets have also been shown to have polynomial time complexity [3].

Due to this series of results, the reachability problem of live and bounded free-choice Petri nets—the problem of deciding if a given marking is reachable from the initial marking—has also been believed to be polynomial since 1991. However, despite some very promising partial results, a proof has remained elusive. In [2] it was shown that the reachability problem of live, bounded and cyclic free-choice Petri nets can be

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* E-mail: esparza@informatik.tu-muenchen.de

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2 Although some results have been extended to slightly larger classes.

3 A Petri net is cyclic if the initial marking is reachable from any other reachable marking.
reduced to the problem of solving a system of $n$ linear equations with $m$ variables, and is therefore polynomial. Later, [3] proved that every reachable marking of a live and bounded free-choice Petri net can be reached by an occurrence sequence of length $O(bm^3)$, where $b$ is the bound of the net, i.e., the maximum number of tokens that a reachable marking can put in a place. More recently, [10] provided a structural characterization of the set of reachable markings which seemed to be an important step towards a polynomial algorithm, and [9] showed how to decide in $O((n + m)^3)$ time whether two given places can be simultaneously marked.

We prove in this paper that, contrary to the expectations raised by all these results, the reachability problem is unlikely to be polynomial. Even the reachability problem for live and safe free-choice Petri nets (where safe means that no reachable marking puts more than one token in any place) is NP-complete.

The paper is organised as follows. Section 2 contains basic definitions. Section 3 contains the NP-completeness proof for live and safe free-choice Petri nets. Section 4 extends the result to the live and bounded ease.

2. Basic definitions

We assume that the reader is familiar with the basic notions and results of the theory of NP-completeness (see [5] for an introduction). We follow the Petri net notations of [3].

A net $N$ is a triple $(S,T,F)$, where $S$ and $T$ are two disjoint, finite sets of places and transitions, and $F \subseteq (S \times T) \cup (T \times S)$ is a flow relation. Places and transitions are generically called nodes. We identify $F$ and its characteristic function $(S \times T) \cup (T \times S) \rightarrow \{0,1\}$.

Given a node $x$ of $N$, $\bullet x = \{ y \mid (y,x) \in F \}$ is the preset of $x$ and $x^* = \{ y \mid (x,y) \in F \}$ is the postset of $x$. Given a set of nodes $X$ of $N$, we define $\bullet X = \bigcup_{x \in X} \bullet x$ and $X^* = \bigcup_{x \in X} x^*$.

Let $T' \subseteq T$. The subnet of $N$ generated by $T'$ is the net $(S', T', F')$, where $S' = \bullet T' \cup T'^*$ and $F' = F \cap ((S' \times T') \cup (T' \times S'))$.

A net $(S,T,F)$ is free-choice if $(s,t) \in F$ implies $\bullet t \times s^* \subseteq F$ for every $s \in S, t \in T$.\footnote{We follow the terminology of [3]. These nets are also called extended free-choice nets in the literature.}

A marking of $N$ is a mapping $M : S \rightarrow \mathbb{N}$. A marking $M$ enables a transition $t$ if $M(s) \geq F(s,t)$ for every place $s$. If $t$ is enabled at $M$, then it can occur, and its occurrence leads to the successor marking $M'$ which is defined for every place $s$ by $M'(s) = M(s) + F(t,s) - F(s,t)$.

A Petri net or system is a pair $(N, M_0)$ where $N$ is a connected net with at least one place and one transition, and $M_0$ is a marking of $N$.

The expression $M \xrightarrow{t} M'$ denotes that the marking $M$ enables transition $t$, and that $M'$ is the marking reached by the occurrence of $t$. The expression $M \xrightarrow{\sigma} M'$, where $\sigma$ is a sequence $\sigma = t_1 t_2 \ldots t_n$ of transitions, denotes that there exist markings
$M_1, M_2, \ldots, M_{n-1}$ such that $M \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \ldots M_{n-1} \xrightarrow{t_n} M'$. Such an expression is called occurrence sequence. We also say that a sequence $\sigma$ of transitions is an occurrence sequence of $(N, M_0)$ if there exists a marking $M$ such that $M_0 \xrightarrow{\sigma} M$.

A marking $M'$ is reachable from $M$ if there exists an occurrence sequence $M \xrightarrow{\sigma} M'$. The reachable markings of a system $(N, M_0)$ are the markings reachable from $M_0$.

A system $(N, M_0)$ is free-choice if $N$ is a free-choice net.

A system is live if for every reachable marking $M$ and every transition $t$ there exists a marking $M'$ reachable from $M$ which enables $t$. A system is $b$-bounded if $M(s) \leq b$ for every place $s$ and every reachable marking $M$, and bounded if it is $b$-bounded for some number $b$. A 1-bounded system is also called safe.

3. Reachability in live and safe free-choice systems

We abbreviate live and safe (bounded) free-choice system to LSFC-system (LBFC-system). We study the following problem:

Reachability (in LSFC-systems)

Given: a LSFC-system $(N, M_0)$, a marking $M$ of $N$;

To decide: is $M$ reachable from $M_0$?

The membership of Reachability in NP is a consequence of the following result [2].

Theorem 3.1 (Short sequence theorem). Let $(N, M_0)$ be a LSFC-system, and let $M$ be a reachable marking. There exists an occurrence sequence $M_0 \xrightarrow{\sigma} M$ such that the length of $\sigma$ is at most

$$\frac{m(m+1)(m+2)}{6},$$

where $m$ is the number of transitions of $N$.

So we can decide in non-deterministic polynomial time if $M$ is a reachable marking.
\((N, M_0)\) is a system and \(T^{>1}, T^{-1}\) are two subsets of transitions of \(N\). The occurrence sequences of \(\mathcal{E}\) are the occurrence sequences of \((N, M_0)\) that contain each transition of \(T^{>1}\) at least once and each transition of \(T^{-1}\) exactly once. A marking \(M\) is reachable in \(\mathcal{E}\) if \(\mathcal{E}\) has an occurrence sequence \(M_0 \xrightarrow{\sigma} M\). Notice that every reachable marking of \(\mathcal{E}\) is a reachable marking of \((N, M_0)\), but not the other way round.

**Input:** A boolean formula \(\phi\) in conjunctive normal form.

**Output:** A constrained system \(\mathcal{E} = (N', M'_0, T^{>1}, T^{-1})\), a marking \(M'\) of \(N'\).

**Specification:**
(a) \((N', M'_0)\) is a LSFC-system;
(b) \(\phi\) is satisfiable if and only if \(M'\) is a reachable marking of \(\mathcal{E}\).

**Input:** The output \(\mathcal{E}, M'\) of the first algorithm.

**Output:** A system \((N, M_0)\), a marking \(M\) of \(N\).

**Specification:**
(a) \((N, M_0)\) is a LSFC-system;
(b) \(M'\) is a reachable marking of \(\mathcal{E}\) if and only if \(M\) is a reachable marking of \((N, M_0)\).

The two algorithms are described in the next two sections.

### 3.1. The first algorithm

As usual, a **literal** is a boolean variable or its negation. A **clause** is a disjunction of literals, and a boolean formula in conjunctive normal form, called **CNF-formula** in the sequel, is a conjunction of clauses. We identify a CNF-formula with the set of clauses that appear in it, and a clause with its set of literals.

Let \(\phi = \{C_1, \ldots, C_m\}\) be a CNF-formula over variables \(x_1, \ldots, x_n\). Without loss of generality, we assume that the set of literals of a clause is neither empty nor the set of all literals.

We construct a constrained system \(\mathcal{E} = (N', M'_0, T^{>1}, T^{-1})\) and a marking \(M'\). The reader may follow the construction in Fig. 1, which shows \(\mathcal{E}\) and \(M'\) for the formula

\[
\phi = x_1 \land (\bar{x}_1 \lor x_2) \land (x_1 \lor \bar{x}_2).
\]

The net \(N'\) is constructed in several steps. We start with the empty net. At each step we add new places, transitions and arcs, or even new subnets. We describe the steps in a rather informal but (hopefully) still precise way.

- For every variable \(x_i\) add to \(N'\) the net \(N_{x_i}\) shown in Fig. 2. The intended meaning of the transition names \(t_{x_i}\) and \(f_{x_i}\) is “set \(x_i\) to true” and “set \(x_i\) to false”, respectively.
- For every clause \(C_j\), add to \(N'\) the net \(N_{C_j}\) shown in Fig. 3. The intended meaning of the transition names \(T_{C_j}\) and \(U_{C_j}\) is “set the truth value of \(C_j\) to true” and “let the truth value of \(C_j\) unchanged”, respectively.
- For each variable \(x_i\) and every clause \(C_j\), connect the net \(N_{x_i}\) to the net \(N_{C_j}\) as shown in Fig. 4, according to four possible cases: (1) both \(x_i\) and \(\bar{x}_i\) appear in \(C_j\); (2) \(x_i\) appears in \(C_j\) but \(\bar{x}_i\) does not; (3) \(\bar{x}_i\) appears in \(C_j\) but \(x_i\) does not; (4) neither \(x_i\) nor \(\bar{x}_i\) appear in \(C_j\). Observe that the connections correspond to the
Fig. 1. Constrained system corresponding to the formula $x_1 \land (\overline{x}_1 \lor y_2) \land (\overline{x}_1 \lor \overline{y}_2)$.

Fig. 2. The net $\mathcal{N}_{x_i}$. 
intended meanings; for instance, if a clause $C_i$ contains $x_i$ but not $\bar{x}_i$, then if $x_i$ is set to true the value of $C_i$ can also be set to true, while if $x_i$ is set to false then the value of $C_i$ must remain unchanged.

- Connect the places $C_1, \ldots, C_m$ to the places $x_1, \ldots, x_n$ by means of auxiliary nodes, as shown in Fig. 5. The intended meaning of transition name $Ax_i$ is "assign to the variable $x_i$ a truth value".

This concludes the construction of $N'$. The marking $M'_0$ puts one token on the place $Start$, and no token anywhere else. We choose $T^> = \{ TC_1, \ldots, TC_m \}$ and $T^\leq = \{ Ax_1, \ldots, Ax_n \}$. Finally, we take $M' = M'_0$. It is obvious that the construction of $N'$ and $M'$ requires only polynomial time.

We briefly explain the intuition behind the construction. Let $\sigma$ be an occurrence sequence of $\sigma'$.  
- The unique occurrence of the transition $Ax_i$ in $\sigma$ signals that $x_i$ is going to be assigned a truth value.
- The nets $Nx_i$ are used to determine the truth values of the variables. Since the transitions $Ax_1, \ldots, Ax_n$ occur exactly once in $\sigma$, for every $1 \leq i \leq n$ either $tx_i$ or $fx_i$ occurs in $\sigma$, but not both. In this way, $\sigma$ determines a unique truth assignment $A_\sigma$ defined by: $A_\sigma(x_i) = true$ if $tx_i$ occurs in $\sigma$, and $A_\sigma(x_i) = false$ if $fx_i$ occurs in $\sigma$.
- After assigning a value to a variable, the sequence $\sigma$ updates the truth values of the clauses according to the new information. The initial truth value of all clauses is $false$. The connections between each pair of nets $Nx_i, NC_i$ guarantee that the occurrence of $TC_i$ in $\sigma$ sets $C_i$ to true, while the occurrence of $UC_i$ leaves its value unchanged. Therefore, $C_i$ is true under $A_\sigma$ if and only if the transition $TC_i$ occurs at least once in $\sigma$.

We have:

**Lemma 3.2.** The first algorithm satisfies its specification.

**Proof.** We consider parts (a) and (b) of the specification separately.

(a) It is immediate to see that $(N', M'_0)$ is free-choice and safe. To prove that it is live, the only (small) difficulty consists of showing that the input place of the transitions
$TC_j$ and $UC_j$ has at least one input transition;\textsuperscript{5} the rest is routine. The input place of a transition $TC_j$ has no input transitions only if for every variable $x_i$ neither $x_i$ nor $\bar{x}_i$ belong to $C_j$, i.e., only if $C_j$ contains no literals at all. The input place of a transition $UC_j$ has no input transitions only if for every variable $x_i$ both $x_i$ and $\bar{x}_i$

\textsuperscript{5} Notice that a $TC_j$ or $UC_j$ transition whose unique input place has no input transition can never occur.
belong to $C_j$, i.e., only if $C_j$ contains all literals. Since by assumption no clause is empty or contains all literals, we are done.

(b) Let $M'_0 \xrightarrow{\alpha} M'$ be an occurrence sequence of $\phi$. Since every transition of $T^{\geq 1}$ occurs in $\sigma$ at least once, the truth assignment $A_{\sigma}$ makes all clauses true, which implies that $\phi$ is satisfiable.

Conversely, assume that $\phi$ is satisfiable. We choose a truth assignment that makes $\phi$ true, and use it to construct an occurrence sequence $M'_0 \xrightarrow{\sigma} M'$ such that every transition of $T^{\geq 1}$ occurs at least once in $\sigma$ and every transition of $T^{\geq 1}$ at least once. The sequence $\sigma$ is the concatenation of sequences $\sigma_1, \ldots, \sigma_n$. Each $\sigma_i$ starts with the occurrence of one of the output transitions of the place $Start$, followed by the corresponding $Ax_i$ transition and the transitions $Fx_i$ or $fX_i$, according to the assignment, and ends with the transition $End$. Due to the way the nets $N_x_i$ and $NC_j$ are connected, $\sigma$ contains every transition of $T^{\geq 1}$ at least once.

3.2. The second algorithm

Let $\phi$ and $M'$ be the output of the first algorithm. We construct an LSFC-system $(N, M_0)$ and a marking $M$, such that $M'$ is a reachable marking of $\phi$ if and only if $M$
is a reachable marking of \((N, M_0)\). In order to define \((N, M_0)\) and \(M\) we need some "building blocks" and a composition operation. The blocks are shown in Figs. 6 and 7.

The following two lemmata are easy to prove, for instance, by inspection of the reachability graph:

**Lemma 3.3.** Let \((N^{\geq 1}, M_0^{\geq 1})\), \(M^{\geq 1}\) and \(t^{\geq 1}\) be as shown in Fig. 6. \((N^{\geq 1}, M_0^{\geq 1})\) is a LSFC-system, and satisfies the following property: there exists an occurrence sequence \(M_0^{\geq 1} \xrightarrow{\sigma} M^{\geq 1}\) containing \(k\)-times the transition \(t^{\geq 1}\) if and only if \(k \geq 1\).

**Lemma 3.4.** Let \((N = 1, M_0 = 1)\), \(M = 1\) and \(t = 1\) be as shown in Fig. 7. \((N = 1, M_0 = 1)\) is a LSFC-system, and satisfies the following property: there exists an occurrence sequence \(M_0 = 1 \xrightarrow{\sigma} M = 1\) containing \(n\)-times the transition \(t = 1\) if and only if \(k = 1\).

The composition operation is defined on (isomorphy classes of) nets in the following way: let \(N_1\) and \(N_2\) be two disjoint nets (if they are not disjoint, rename places and transitions appropriately), and let \(t_1\) and \(t_2\) be transitions of \(N_1\) and \(N_2\), respectively.

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*In fact, a stronger statement holds: \(M_0 = 1 \xrightarrow{\sigma} M = 1\) if and only if \(\sigma = 1 = t = 1\).*
The *merge* of $t_1$ and $t_2$ is the operation consisting of the following three parts:\footnote{We give an informal definition, which we consider to be precise enough for our purposes. A more formal definition would just be more difficult to read.}

- put $N_1$ and $N_2$ side by side;
- remove $t_1$ and $t_2$ together with their incident arcs;
- add a new transition $t$; let the preset (postset) of $t$ be the union of the presets (postsets) of $t_1$ and $t_2$.

Let $N$ be the net obtained after performing this operation. The set of places of $N$ is the disjoint union of the places of $N_1$ and $N_2$. Therefore, a marking of $N$ is characterised by its projections onto these two sets of places. We denote by $(M_1, M_2)$ the marking that projects onto markings $M_1$ of $N_1$ and $M_2$ of $N_2$.

The composition operation is extended to net systems as follows: the system obtained after the merge of transitions $t_1$ and $t_2$ of the systems $(N_1, M_1)$ and $(N_2, M_2)$ is $(N, (M_1, M_2))$, where $N$ is the net defined above.

We are now ready to construct the system $(N, M_0)$. Start with $(N', M'_0)$, and merge each transition of $T^{>1}$ with the transition $t^{>1}$ of a fresh copy of $(N^{>1}, M_0^{>1})$. Then, merge each transition of $T^{=1}$ with the transition $t^{=1}$ of a fresh copy of $(N^{=1}, M_0^{=1})$.

The system $(N, M)$ (and with it the marking $M$) is constructed analogously: just substitute $M$ for $M_0$, $M^{>1}$ for $M_0^{>1}$, and $M^{=1}$ for $M_0^{=1}$.

At this point, the reader is possibly willing to accept the truth of the following lemma without further discussion. If this is not the case, a (rather tedious) proof is given in the appendix.

**Lemma 3.5.** The second algorithm satisfies its specification.

We can now easily prove NP-hardness, and, using the result at the beginning of the section, NP-completeness of Reachability:

**Theorem 3.6.** Reachability is NP-complete.

**Proof.** Membership in NP was shown at the beginning of this section. NP-hardness follows immediately from Lemmas 3.2 and 3.5, which together reduce the satisfiability problem of CNF-formulas to Reachability. □

4. Reachability in live and bounded free-choice systems

We show that the reachability problem of LBFC-systems, not necessarily safe, is still NP-complete. Clearly, the problem is NP-hard, and so it suffices to prove membership in NP. In [3], Desel and the author prove a generalisation of the short sequence theorem (Theorem 3.1) to $b$-bounded systems: if $M$ is reachable from $M_0$, then there exists an
occurrence sequence $M_0 \xrightarrow{\sigma} M$ such that the length of $\sigma$ is at most

$$b \frac{m(m+1)(m+2)}{6},$$

where $m$ is the number of transitions of $N$.

It follows from this generalisation that the reachability problem of live and $b$-bounded free-choice systems belongs to NP for every $b \geq 1$. However, it does not follow that the reachability problem for LBFC-systems belongs to NP: the reason is that a LBFC-system encoded into a binary string of length $k$ can be $O(2^k)$-bounded. In order to prove membership in NP for this problem, we use a result due to Lee et al. [10]. The result makes use of the so-called traps. A trap of a net $(S, T, F)$ is a subset of places $S'$ such that $S' \subseteq S$.

**Theorem 4.1.** Let $(N, M_0)$ be a LBFC-system, where $N = (S, T, F)$. A marking $M$ is reachable from $M_0$ if there exists a vector $X \in \mathbb{N}^{|T|}$ such that

1. the linear equation $M(s) = M_0(s) + \sum_{t \in T} (F(t,s) - F(s,t)) \cdot X(t)$ holds for every place $s \in S$, and
2. every trap of the subnet of $N$ generated by the transitions $t \in T$ satisfying $X(t) > 0$ is marked under $M$.

This theorem immediately leads to the following nondeterministic polynomial time algorithm:

1. Guess a subset $T'$ of transitions of $N$ ($T'$ will be the set of transitions $t$ for which $X(t) > 0$).
2. Check that every trap of the subnet generated by $T'$ is marked at $M$ (a polynomial algorithm for this problem can be found in [11, 3]).
3. For each transition $t$, if $t \in T$ then add $X(t) \geq 1$ to the equations of (1); otherwise, add $X(t) = 0$. Check in nondeterministic polynomial time that the resulting equation system has a solution in the natural numbers (for a proof of the fact that nondeterministic polynomial time suffices see for instance [7, pp. 336–340]).

It should be remarked that the proof of Theorem 4.1 given in [10] is very complicated, and not well understood by many people (including the author). Therefore, the theorem should be used with a bit of care before a more transparent proof is found.

5. Conclusions

We have determined the complexity of reachability in live and safe and live and bounded free-choice systems, a problem which had been open for several years. Contrary to the expectations, reachability turns out to be NP-complete. The NP-hardness proof is a rather straightforward reduction from the satisfiability problem for boolean formulas in conjunctive normal form. The author now believes that the problem was open for such a long time not because of its difficulty, but because the researchers interested in it (including the author) directed their efforts in the wrong direction.
the other hand, these efforts have produced many of the nice results on reachability in free-choice systems mentioned in the introduction.

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Appendix. Proof of Lemma 3.5

We need two preliminary lemmata. The first one is an easy consequence of the definition of the merge operation:

Lemma A.1. Let \( N \) be the net obtained after the merge of transitions \( t_1 \) and \( t_2 \) of the nets \( N_1 \) and \( N_2 \). \( (L_1, L_2) \xrightarrow{a} (L'_1, L'_2) \) is an occurrence sequence of \( N \) iff there exist two occurrence sequences

\[
L_1 \xrightarrow{\sigma_1^1 \cdots \sigma_1^i \cdots \sigma_1^n} L'_1 \quad \text{and} \quad L_2 \xrightarrow{\sigma_2^j \cdots \sigma_2^2 \cdots \sigma_2^n} L'_2
\]

of \( N_1 \) and \( N_2 \), respectively, such that \( \sigma = \tau^1 \tau^2 \tau^3 \cdots \tau^n \), where \( \tau^i \) is an arbitrary interleaving of \( \sigma_1^i \) and \( \sigma_2^i \).

We have then:

Lemma A.2. Let \( (N_1, M_1) \) and \( (N_2, M_2) \) be LSFC-systems, and let \( t_1 \) and \( t_2 \) be transitions of \( N_1 \) and \( N_2 \), respectively.

(1) The system \( (N, M) \) obtained after the merge of \( t_1 \) and \( t_2 \) is live and safe.

(2) If \( N_1, N_2 \) are free-choice nets, \( (t_1^*) = \{t_1\} \) in \( N_1 \) and \( (t_2^*) = \{t_2\} \) in \( N_2 \), then \( N \) is also free-choice.

Proof. (1) It follows from Lemma 6.1 that \( (L_1, L_2) \) is a reachable marking of \( (N, M) \) if and only if \( L_1 \) and \( L_2 \) are reachable markings of \( (N_1, M_1) \) and \( (N_2, M_2) \). Since \( (N_1, M_1) \) and \( (N_2, M_2) \) are safe, \( (N, M) \) is safe.

For the liveness part, let \( (L_1, L_2) \) be an arbitrary reachable marking of \( (N, M) \), and let \( u \) be an arbitrary transition of \( N \). Consider two cases:

- \( u = t \) (i.e., \( u \) is the merge of \( t_1 \) and \( t_2 \)). Since \( (N_1, M_1) \) and \( (N_2, M_2) \) are live, there exist occurrence sequences \( L_1 \xrightarrow{\sigma_1^1} L'_1 \) and \( L_2 \xrightarrow{\sigma_2^1} L'_2 \). Let \( \tau \) be an arbitrary interleaving of \( \sigma_1 \) and \( \sigma_2 \). Then \( (L_1, L_2) \xrightarrow{\tau} (L'_1, L'_2) \) is an occurrence sequence of \( N \).
• \( u \neq t \). Assume, without loss of generality, that \( u \) belongs to \( N_1 \). Since \((N_1, M_1)\) and \((N_2, M_2)\) are live, there exist occurrence sequences

\[
\begin{align*}
L_1 & \xrightarrow{\sigma_1^1, \sigma_2^1, \ldots, \sigma_1^u} L_1' \\
L_2 & \xrightarrow{\sigma_2^1, \sigma_3^2, \ldots, \sigma_2^2} L_2'
\end{align*}
\]

Let \( \tau' \) be an arbitrary interleaving of \( \sigma_1^1 \) and \( \sigma_2^2 \). Then

\[
(L_1, L_2) \xrightarrow{\tau_1^1 \tau_2^2 \ldots \tau_1' u} (L_1', L_2')
\]

is an occurrence sequence of \( N \).

(2) Follows immediately from the definitions. \( \square \)

We are now ready to prove Lemma 3.5.

**Lemma 3.5** (Restated). The second algorithm satisfies its specification.

**Proof.** Let \( \mathcal{C} \) and \( M' \) be the output of the first algorithm for a given input, and let \((N, M_0)\) and \( M \) be the output of the second algorithm. We prove parts (a) and (b) of the specification separately.

(a) We show that \((N, M_0)\) is a LSFC-system. Liveness and boundedness of \((N, M_0)\) follows from iteration of application of Lemma A.2(1). To prove free-choiceness we apply Lemma 6.2(2): \( N' \), \( N^{=1} \) and \( N^{-1} \) are free choice, every transition \( t \in T^{\geq 1} \cup T^{=1} \) satisfies \((^* t)^* = \{ t \}\) in the net \( N' \), and the transitions \( t^{\geq 1} \) and \( t^{=1} \) satisfy the same condition in the nets \( N^{\geq 1} \) and \( N^{=1} \), respectively.

(b) We show that \( M' \) is a reachable marking of \( \mathcal{C} \) if and only if \( M \) is a reachable marking of \((N, M_0)\).

Assume that \( \mathcal{C} \) has an occurrence sequence \( M_0^{\mathcal{C}} \xrightarrow{\tau} M' \). By Lemma 3.3, for each transition \( t \in T^{\geq 1} \) there exists an occurrence sequence \( M_0^{\geq 1} \xrightarrow{\sigma^{=1}} M^{=1} \) which contains the transition \( t^{=1} \) exactly \( \sigma(t) \) times. By Lemma 3.4, there exists an occurrence sequence \( M_0^{=1} \xrightarrow{\sigma^{=1}} M^{=1} \) which contains the transition \( t^{=1} \) exactly once. By repeatedly applying Lemma 6.1 we obtain from these sequences an occurrence sequence of \((N, M_0)\) leading to \( M \).

Conversely, assume that there exists an occurrence sequence \( M_0 \xrightarrow{\sigma} M \) in \((N, M_0)\). For each \( t \in T^{=1} \) \((t \in T^{=1})\), let \( \sigma_t \) be the projection of \( \sigma \) onto the transitions of the fresh copy of \( N^{=1} \) \((N^{=1})\) corresponding to \( t \). By Lemma A.1 we have \( M_0^{=1} \xrightarrow{\sigma_t} M^{=1} \). By Lemma 3.4 (Lemma 3.3), \( \sigma_t \) contains the transition \( t^{=1} \) exactly once (the transition \( t^{\geq 1} \) more than once). Then, the projection of \( \sigma \) onto the transitions of \( N \) yields an occurrence sequence of \( \mathcal{C} \) leading to \( M' \). \( \square \)

**References**


