Transfinite mean value interpolation

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Transfinite interpolation

Given \( \Omega \subset \mathbb{R}^2 \), a convex or non-convex set, possibly with holes.

**Lagrange transfinite interpolation**
We are given \( f : \partial \Omega \to \mathbb{R} \).
Find \( g : \Omega \to \mathbb{R} \) that interpolates \( f \) on \( \partial \Omega \).

**Hermite transfinite interpolation**
We are given \( f : \partial \Omega \to \mathbb{R} \) and \( \frac{\partial f}{\partial n} : \partial \Omega \to \mathbb{R} \).
Find \( g : \Omega \to \mathbb{R} \) that interpolates \( f \) and \( \frac{\partial g}{\partial n} \) matches \( \frac{\partial f}{\partial n} \) on \( \partial \Omega \).

- Lagrange can be solved by solving the harmonic equation
- Hermite can be solved by solving the biharmonic equation

\[ \longrightarrow \text{ But we want something simpler...} \]

Let

- \( x \) be a point inside the convex set \( \Omega \);
- \( Q(x, \theta) \) be the **infinite** line through \( x \) in the direction \( \theta \).
- Let \( p_1(x, \theta) \) and \( p_2(x, \theta) \) be the two intersections between \( Q(x, \theta) \) and \( \partial \Omega \),

then we define

\[
g_{GW}(x) = \frac{1}{2\pi} \int_0^{2\pi} \text{lerp} \left( f(p_1(x, \theta)), f(p_2(x, \theta)), \frac{\|p_1(x, \theta) - x\|}{\|p_1(x, \theta) - p_2(x, \theta)\|} \right) d\theta.
\]

- Works only for convex sets.
- Evaluation requires numerical integration

\[\text{\rightarrow must find intersections for each integration point!}\]
A mean value approach

Let

- \( L(x, \theta) \) be the semi-infinite line originating at \( x \) in the direction \( \theta \).
- \( p(x, \theta) \) be the intersection of \( L(x, \theta) \) and \( \partial \Omega \).

and define the “radially linear” function \( F \) as

\[
F(x + r(\cos \theta, \sin \theta)) = \text{lerp} \left( g(x), f(p(x, \theta)), \frac{r}{\|p(x, \theta) - x\|} \right).
\]
We want $F$ to satisfy the Mean Value property at $x$.
Let $\Gamma$ be any circle at $x$ with radius $r$, then

$$F(x) = \frac{1}{2\pi r} \int_{\Gamma} F(z) \, dz,$$

whose unique solution is

$$g(x) = \frac{1}{\phi(x)} \int_{0}^{2\pi} \frac{f(p(x, \theta))}{\|p(x, \theta) - x\|} \, d\theta, \quad \phi(x) = \int_{0}^{2\pi} \frac{1}{\|p(x, \theta) - x\|} \, d\theta,$$

which is the “angle integral” formula for the MV interpolant $g$.

- Generalizes to non-convex sets
- Evaluation still requires numerical integration.
- Still must find an intersection for each integration point!
- How do we differentiate this thing?
- Luckily, we have two other formulas...
The boundary integral formula [Ju, Schaefer, Warren 2005]

Suppose $\mathbf{c} : [a, b] \to \mathbb{R}^2$ is an anticlockwise representation of $\partial \Omega$.

Then

$$
\frac{d\theta}{dt} = \frac{(\mathbf{c}(t) - \mathbf{x}) \times \mathbf{c}'(t)}{\|\mathbf{c}(t) - \mathbf{x}\|^2}
$$

which gives

$$
\phi(\mathbf{x}) = \int_a^b \frac{(\mathbf{c}(t) - \mathbf{x}) \times \mathbf{c}'(t)}{\|\mathbf{c}(t) - \mathbf{x}\|^3} dt,
$$

and

$$
g(\mathbf{x}) = \frac{1}{\phi(\mathbf{x})} \int_a^b \frac{(\mathbf{c}(t) - \mathbf{x}) \times \mathbf{c}'(t)}{\|\mathbf{c}(t) - \mathbf{x}\|^3} f(\mathbf{c}(t)) dt.
$$
The polygonal formula

Suppose $\Omega$ is a polygon with vertices $p_1, \ldots, p_n$.

Then

$$g(x) = \frac{1}{\phi(x)} \sum_i w_i(x) f(p_i),$$

where

$$\phi(x) = \sum_i w_i(x),$$

and

$$w_i(x) = \frac{\tan(\alpha_{i-1}(x)/2) + \tan(\alpha_i(x)/2)}{\|p_i - x\|}. $$
We have three formulations for the MV interpolant:

- **The polygonal formula:**
  - closed form
  - easy to find expressions for derivatives.

- **The boundary integral formula**
  - needs adaptive numerical quadrature for evaluation.
  - easy to find expressions for derivatives.

- **The angle integral**
  - describes the interpolant along a particular direction.
The MV weight function

A lot of properties can be deduced from the “weight function” $\psi$

\[
\psi(x) = \frac{1}{\phi(x)} = 1 \left/ \int_0^{2\pi} \frac{1}{\|p(x, \theta) - x\|} d\theta, \right.
\]

\[
= 1 \left/ \int_a^b \frac{(c(t) - x) \times c'(t)}{\|c(t) - x\|^3} dt, \right.
\]

\[
= 1 \left/ \sum_i \frac{\tan(\alpha_{i-1}(x)/2) + \tan(\alpha_i(x)/2)}{\|p_i - x\|}. \right.
\]
Minimum principle for $\psi$

For arbitrary $\Omega$, we have that

$$\Delta \phi(x) = 3 \int_0^{2\pi} \sum_{j=1}^{n(x, \theta)} \frac{(-1)^{j-1}}{\|p_j(x, \theta) - x\|^3} \, d\theta$$

from which follows that

- $\psi$ has no local minima in $\Omega$.  

$$\Delta \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$
Bounds on $\psi$

For all $x \in \Omega$ we have that

$$\frac{1}{2\pi} \text{dist}(x, \partial \Omega) \leq \psi(x) \leq c \text{dist}(x, \partial \Omega),$$

- $c$ depends on $\text{dist}(M_E, \partial \Omega)$, the distance between $\partial \Omega$ and its exterior medial axis
- If $\Omega$ is convex, then $c = \frac{1}{2}$.

$\Rightarrow$ For all $x \in \Omega$, $\psi > 0$.

The plot shows the upper and lower bounds and $\psi$ along a cross-section when $\Omega$ is the unit disc.
Proof of interpolation for the Lagrange MV interpolant

If

- $f$ is continuous,
- $\partial \Omega$ and any line intersects a bounded number of times,
- and $\text{dist}(M_E, \partial \Omega) > 0$

then

- $g$ interpolates $f$. 
Normal derivatives of $\psi$ and $g$

If
\[
\text{dist}(M_E, \partial \Omega) > 0 \quad \text{and} \quad \text{dist}(M_I, \partial \Omega) > 0,
\]
then, for all $y \in \partial \Omega$,

- the inward normal derivative for $\psi$ is
  \[
  \frac{\partial \psi}{\partial n}(y) = \frac{1}{2}
  \]

- the inward normal derivative for the Lagrange interpolant $g$ is
  \[
  \frac{\partial g}{\partial n}(y) = \frac{1}{2} \int_a^b \frac{(c(t) - y) \times c'(t)}{\|c(t) - x\|^3} (f(c(t)) - f(y)) \, dt.
  \]
Hermite mean value interpolation

In one variable, we have the problem

\[ p(x_i) = f(x_i) \quad \text{and} \quad p'(x_i) = f'(x_i), \quad i = 0, 1. \]

One approach of expressing \( p \) is

\[ p(x) = g_0(x) + \psi(x)g_1(x), \]

where

- \( g_0 \) and \( g_1 \) are Lagrange interpolants,
- \( \psi \) vanishes at \( x_0 \) and \( x_1 \) and \( \psi' \) is nonzero at \( x_0 \) and \( x_1 \).

Which gives the conditions

\[ g_0(x_i) = f(x_i) \quad \text{and} \quad g_1(x_i) = \frac{f'(x_i) - g_0'(x_i)}{\psi'(x_i)}. \]
In two variables, we can generalize a similar problem,

\[ p(y) = f(y) \quad \text{and} \quad \frac{\partial p}{\partial n}(y) = \frac{\partial f}{\partial n}(y), \quad y \in \partial \Omega. \]

and let \( p \) be on the form

\[ p(x) = g_0(x) + \psi(x)g_1(x). \]

We can use the MV-\( \psi \) since

\[ \psi(y) = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial n}(y) = \frac{1}{2}, \quad y \in \partial \Omega, \]

and let \( g_0 \) and \( g_1 \) be MV Lagrange interpolants.
Then, for \( y \in \partial \Omega \) we get the conditions

\[
g_0(y) = f(y)
\]

\[
g_1(y) = \left( \frac{\partial f}{\partial n}(y) - \frac{\partial g_0}{\partial n}(y) \right) / \frac{\partial \psi}{\partial n}(y).
\]
Application: Smooth mappings

Reference shape  Computational domain

(MV-Lagrange) 

Conjecture: Lagrange interpolation from convex sets to convex sets is always injective.
Application: WEB-splines [Höllig, Reif, Wipper 2001]

Idea: Use $\psi$ as a weight function for WEB-splines

- Parametric circle
- Two nested ellipses
- Polygon
- Piecewise cubic Bézier curve
Solution to Poisson’s equation using bicubic web-splines

Using implicit weight function

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Using MV weight function

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Inhomogeneous Poisson’s equation

True solution

MV Lagrange interpolant

Homogeneous solution

Inhomogeneous solution
Conclusions

▶ The **Lagrange mean value interpolant** does in fact interpolate.
▶ Constructed a **Hermite mean value interpolant**.
▶ The **mean value weight function** has nice properties:
  ▶ positive;
  ▶ $C^\infty$-smooth;
  ▶ bounded by the distance function:
    $\mapsto$ a very smooth distance-like function without ridges along the inner medial axis!
  ▶ constant normal derivate;
  ▶ has no local minima in $\Omega$;
▶ The mean value constructions are relatively easy to compute:
  ▶ The polygonal case has a closed form.
  ▶ The boundary integral must be calculated numerically, but:
    ▶ Strong influence of the boundary region closest to the point of evaluation.
    $\mapsto$ Adaptivity pays off.
  ▶ Simpler than solving a PDE.
Thank you for listening!

Questions?