Matrices of zeros and ones with given line sums and a zero block

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Plan:

- background, motivation
- the main results
- algorithm
- extensions

Remark:

- existence of (0,1)-matrices with certain properties
- main be reformulated in terms of bipartite graphs
Motivation/The problem:

Feasible set in the transportation problem:

\[
\begin{align*}
\sum_{j=1}^{n} x_{i,j} &= a_i \quad (i \leq m) \\
\sum_{i=1}^{m} x_{i,j} &= b_j \quad (j \leq n) \\
&\quad x_{i,j} \geq 0 \quad (i \leq m, j \leq n).
\end{align*}
\]

Feasible iff \( \sum_i a_i = \sum_j b_j \).

More complicated versions:

- \( 0 \leq x_{i,j} \leq 1 \)
- \( x_{i,j} \in \{0, 1\} \)
- some \( x_{i,j} \)'s are fixed to zero

The (first) problem: characterize the 
existence of \((0, 1)\)-matrices (of size \(m \times n\))
with given line sums and a zero block (of fixed size).
The matrix

\[ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & O \end{bmatrix} \]

where the block \( A_1 \) has size \( p \times q \). Here \( p \leq m \) and \( q \leq n \).

Matrix class: \( \mathcal{A}_{p,q}(R, S) \): row sum vector \( R \) and column sum vector \( S \).

Example: \( R = (5, 5, 5, 4, 4, 3, 2, 2) \), \( S = (5, 5, 5, 3, 4, 3, 3, 2) \), \( p = 5 \) and \( q = 4 \).

\[
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\]

Motivation:

- Nice results known without the zero block: Gale-Ryser theorem
- Discrete tomography: discrete 2-dim. images, projections, known zero pattern.
Known results, when $p = m$, $q = n$ (no zero block)

We may assume $R$ and $S$ nonincreasing (monotone).

**Theorem. Equivalent:**

(i) $A_{m,n}(R, S)$ is nonempty.

(ii) $S < R^*$ (Gale-Ryser theorem), i.e.,

\[
\sum_{j=1}^k s_j \leq \sum_{j=1}^k r_j^* \quad (k \leq n - 1)
\]

\[
\sum_{j=1}^n s_j = \sum_{j=1}^n r_j^*.
\]

(iii) *The structure matrix $T$ is nonnegative.*

Conjugate vector: $r_j = (R^*)_j = |\{i : r_i \geq k\}|$.

Ryser’s structure matrix $T = [t_{kl}]$: for

$(0 \leq k \leq m, \ 0 \leq l \leq n)$

\[
t_{k,l} = kl + \sum_{i=k+1}^m r_i - \sum_{j=1}^l s_j
\]
New results: general $p$ and $q$.

Define the structure matrix for the class $\mathcal{A}_{p,q}(R, S)$.

For $0 \leq k \leq p$ and $0 \leq l \leq q$ let

$$t_{k,l} = \tau + kl - \sum_{i=1}^{k} r_i - \sum_{j=1}^{l} s_j$$

$$- \sum_{i=p+1}^{m} (r_i - l)^+ - \sum_{j=q+1}^{n} (s_j - k)^+.$$

Here $\tau = \sum_{i} r_i = \sum_{j} s_j$. $T$ has size $(p + 1) \times (q + 1)$.

Specializes into Ryser’s matrix when $p = m$, $q = n$.

**Theorem.**

$\mathcal{A}_{m,n}(R, S)$ is nonempty if and only if the structure matrix $T$ is nonnegative.
Remarks:

- Concerning the proof: May be shown using the maxflow-mincut theorem to a suitable network, and analyze the structure of minimum cuts.

- This “cut reduction” is essential as there are an exponential number of cut constraints

- $T$ may be calculated efficiently due to a difference property:

$$t_{k+1,l} - t_{k,l} = t_{k+1,l+1} - t_{k,l+1} - 1$$

So, $T$ is determined by its first row and column.

- The nonnegativity of $T$ may be interpreted combinatorially: one may give a lower bound on the number of ones in each of the three nonzero blocks of the matrix. This bound must be $\leq \tau$, and this corresponds to $T \geq O$. 
Let $m = n = 10$, $p = q = 7$, 
$R = (9, 8, 6, 6, 5, 5, 4, 3, 3, 3)$, and 
$S = (7, 6, 6, 6, 6, 6, 4, 3, 2)$.

Then the structure matrix is

$$
T = \begin{pmatrix}
34 & 30 & 27 & 24 & 18 & 12 & 6 & 0 \\
28 & 25 & 23 & 21 & 16 & 11 & 6 & 1 \\
23 & 21 & 20 & 19 & 15 & 11 & 7 & 3 \\
19 & 18 & 18 & 15 & 12 & 9 & 6 \\
14 & 14 & 15 & 16 & 14 & 12 & 10 & 8 \\
9 & 10 & 12 & 14 & 13 & 12 & 11 & 10 \\
4 & 6 & 9 & 12 & 12 & 12 & 12 & 12 \\
0 & 3 & 7 & 11 & 12 & 13 & 14 & 15 \\
\end{pmatrix}.
$$

Here $T \geq O$ so $\mathcal{A}_{7,7}(r, s)$ is nonempty.

Modification: let $p = q = 5$ and maintain $R$ and $S$. Then $t_{5,0} = -2$ so $\mathcal{A}_{5,5}(R, S)$ is empty.
Algorithms

Ryser’s algorithm: constructs a \((0, 1)\)-matrix with row sum \(R\) and column sum \(S\) (no zero block constraint):

- Starts with the “maximal matrix” \(\bar{A}\) with row sum \(R\) and ones left justified, i.e., column sum is \(R^*\).

- Shift the last 1 in certain rows of \(\bar{A}\) to column \(n\) in order to achieve the sum \(s_n\).

- The 1’s in column \(n\) are to appear in those rows in which \(\bar{A}\) has the largest row sums, giving preference to the bottommost positions in case of ties.

- Proceed inductively to construct columns \(n - 1, \ldots, 2, 1\).

This works precisely when \(S < R^*\).
Assume: $R$ monotone, $q$ an integer with $1 \leq q \leq n$, and $s_{q+1} \geq s_{q+2} \geq \cdots \geq s_n$. Let
\[ A = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \]
be a matrix in $\mathcal{A}(R, S)$, where $A_1$ has size $m \times q$.

The modified Ryser algorithm: use Ryser’s algorithm to construct columns $n, n - 1, \ldots, q + 1$. This gives a matrix
\[ \hat{A} = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \end{bmatrix} \]
where the row sums of the $m$ by $k$ submatrices of $\hat{A}$ formed by the first $k$ columns are monotone for $k = q, q + 1, \ldots, n$.

The matrix $\hat{A}_2$ is uniquely determined by $R$, $S$, and $q$, and we denote it by $\hat{A}^q$. It is called canonical column $q$-submatrix relative to $R$ and $S'_q$. Here $S'_q = (s_{q+1}, s_{q+2}, \ldots, s_n)$. 
Theorem.

Assume that $A_{p,q}(R,S) \neq \emptyset$. Then there exists a matrix

$$\tilde{A}_{p,q} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}^q \\ p\tilde{A} & O \end{bmatrix}$$

in the class $A_{p,q}(R,S)$ such that $\tilde{A}^q$ is the canonical column $q$-submatrix relative to $R_p$ and $S'_q$, and $p\tilde{A}$ is the canonical row $p$-submatrix relative to $R'_p$ and $S_q$, and $\tilde{A}_1$ is the canonical matrix in the matrix class to which it belongs.
From this we get a computationally simple algorithm to construct a matrix $A$ in $\mathcal{A}_{p,q}(R,S)$:

(a) Use the modified Ryser algorithm to construct the canonical column $q$-submatrix $\tilde{A}^q$ relative to $R_p$ and $S_q'$. Let the row sum vector of $\tilde{A}^q$ be $\tilde{R}_p$.

(b) Use the modified Ryser algorithm to construct the canonical row $p$-submatrix $p\tilde{A}$ relative to $S_q$ and $R_p'$. Let the column sum vector of $p\tilde{A}$ be $\tilde{S}_q$.

(c) Use the Ryser algorithm to construct the canonical matrix $\tilde{A}$ in the class $\mathcal{A}(R_p - \tilde{R}_p, S_q - \tilde{S}_q)$.

(d) Let

$$\tilde{A}_{p,q} = \begin{bmatrix} \tilde{A} & \tilde{A}^q \\ p\tilde{A} & O \end{bmatrix}.$$
Let $R = (5, 5, 5, 4, 4, 3, 2, 2)$ and $S = (5, 5, 5, 3, 4, 3, 3, 2)$, and let $p = 5$ and $q = 4$. The matrix in $\mathcal{A}_{5,4}(R, S)$ constructed by the algorithm is

$$
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$
What is the rank of $T$?

**Theorem.**

Consider the structure matrix $T$. Then

$$1 \leq \text{rank}(T) \leq 3.$$

Moreover the rank is 1 iff ..., and the rank is 2 iff ....

Here ... and ... may be given combinatorial interpretations!

Extensions.

More general set where zeros are fixed. New class: $\mathcal{A}_P(R, S)$.

(i) $P_j$ permitted positions of 1’s in column $j$:

$$P_1 \supseteq P_2 \supseteq \cdots \supseteq P_n.$$

(ii) Special case (Young-pattern): $P$ is specified by a vector $K = (k_1, k_2, \ldots, k_n)$ satisfying $k_1 \geq k_2 \geq \cdots \geq k_n$ and for each $j \leq n$

$$P_j = \{1, 2, \ldots, k_j\}.$$

$$A = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & \mathbf{0} & \mathbf{0} \\
0 & 1 & 1 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix}.$$
Lemma. \( \mathcal{A}_P(R, S) \) contains a matrix \( \hat{A} \) where the 1’s in column \( n \) appear in those rows in \( P_n \) in which \( r_i \) is largest, giving preference to the bottommost positions in case of ties.

Theorem. \( \mathcal{A}_P(R, S) \) contains a unique matrix \( \hat{A} \) such that for each \( k \leq n \) the submatrix \( A_k \) consisting of the first \( k \) columns of \( A \) is a last-column canonical matrix in its class.

Theorem. Let \( A \) and \( B \) be two given matrices in \( \mathcal{A}_P(R, S) \). Then there is a sequence of interchanges that transforms \( A \) to \( B \) with every intermediary matrix in \( \mathcal{A}_P(R, S) \).
Generalized Ryser algorithm:

1. (Initialize) Let \( k = n \) and let \( \hat{R} = R \).

2. (Determine column \( k \)) Find the indices in \( P_k \) corresponding to the \( s_k \) largest positive components of \( \hat{R} \) where we prefer largest indices in case of ties. Let the \( k' \)th column of \( \tilde{A} \) have ones in the \( s_k \) positions just found.

3. (Update row sum) Let \( \hat{R} = R - \tilde{A}(\cdot, k) \) (the row sum vector after the last column has been deleted). If \( k > 1 \), reduce \( k \) by 1 and go to Step 2.