Estimation of Scalar Ocean Wave Spectra by the Maximum Entropy Method

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Abstract—This paper describes the estimation of ocean wave spectra with the maximum entropy method. This method is based on an autogressive model for the time series of sea surface elevation, and gives an all-pole estimate. The wave series are also prefiltered to introduce one zero in the spectral estimate. The methods are successfully applied to estimate the spectra of some typical wave series, requiring between 10 and 30 parameters.

Our comparison with conventional spectral estimates also show that the new methods give good estimates of the spectral moments, that is important because the moments are parameters in a statistical description of the sea surface. The algorithms involved are well suited for real-time estimation of high-resolution wave spectra.

INTRODUCTION

DESCRIPTION of ocean waves is of vital importance for any engineering activity in the ocean environment, and therefore a considerable effort is spent on the collection and analysis of wave data. Considering the importance and the effort involved it therefore seems natural also to investigate alternative methods for collection and processing wave data. With the large amounts of data involved, it is also worthwhile to investigate if it is possible to reduce this amount without reducing the information content significantly.

The purpose of this paper is to present the results of an attempt to apply parametric methods to the problem of estimating scalar ocean wave spectra. The method used is known as the maximum entropy method and is based on an autoregressive model for the direct or prefiltered time series of wave heights. This method has been used by statisticians for a long time to characterize time series. Its ability to give high-resolution spectral estimates, however, has only been explored in the last decade or so. This application is often attributed to Burg [3]. Several papers have described various applications of the maximum entropy method. Among them are the analysis of seismic signals [6], speech [7], and EEG signals. Compared to the conventional periodogram and correlogram spectral estimates, the maximum entropy method generally gives higher resolution and a description of the spectrum with fewer parameters [5].

This paper first gives a brief outline of the theory behind the maximum entropy method and the conventional methods. To gain more insight into the properties of the methods the underlying autoregressive and moving average models are discussed.

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In the second part examples of different wave series are analyzed. The spectra are found using the autoregressive model on either the raw time series or the time series prefiltered with one moving average parameter. These spectral estimates are then compared to conventional smoothed periodogram estimates. The first 4 moments of the spectra have also been calculated. Finally, a system for real-time processing and data reduction with a microprocessor in the wave-measuring buoy is proposed.

THE MAXIMUM ENTROPY PRINCIPLE

The power spectrum of a stationary process is the discrete Fourier transform of its autocorrelation function as given by

$$P(f) = T \sum_{k=-\infty}^{\infty} R(kT) \exp(-2\pi jkTf) \tag{1}$$

where T is the sampling interval.

It is useful to compare various spectral estimates by looking at the implied assumptions about the autocorrelation function.

In the correlogram method the infinite sum in (1) is truncated. To smooth out some of the effects of this the q estimates of the autocorrelation function are often multiplied by a window function $w(k)$. The new estimate is given by

$$S(f) = T \sum_{k=-q}^{q} \hat{R}(kT) w(k) \exp(-2\pi jkTf). \tag{2}$$

Another way to overcome the effects of truncation would be to extrapolate the autocorrelation function from p estimated values, where p often is less than q. The two concepts are illustrated in Fig. 1. In this figure the amplitude of the windowed estimate decreases gradually to zero as the lag reaches q. The extrapolated estimate, however, is nonzero beyond both p and q.

Maximum entropy spectral analysis is one way to achieve this extrapolation according to the following criteria:

1) The spectral estimate shall agree with the p estimated values of the autocorrelation function.

2) The estimate shall be based on the fewest possible assumptions about the values of the autocorrelation function beyond lag p (maximizing the entropy).

When treating these criteria mathematically the result is an all-pole spectral estimate [4]

$$S(f) = \frac{b_0{}^2 T}{\left| 1 + \sum_{i=1}^{p} a_i \exp(-2\pi jiTf) \right|} \tag{3}$$

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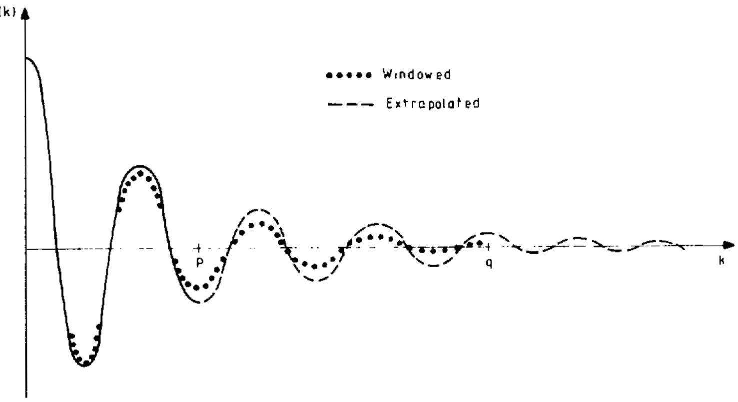

Fig. 1. Various ways of obtaining a useful estimate of the autocorrelation function.

where $b_0, a_1, \cdots, a_p$ are unknown parameters. Intuitively this seems to be a better approximation to the infinite sum in (1) than the correlogram estimate.

It should be pointed out that the periodogram spectral estimate is often used instead of the correlogram because it is computationally cheaper. It is given as

$$S(f) = \frac{T}{N} \sum_{m=0}^{M-1} \left| \sum_{n=0}^{N_1-1} x(n+m \cdot N_1) \exp(-2\pi jnTf) \right|^2. \quad (4)$$

The time series, $x(n)$ of length N, is divided into M series of length $N_1 = N/M$. Each series is Fourier transformed and magnitude squared. The spectral estimate is obtained by averaging these M periodograms, and has $2M$ degrees of freedom.

Both the correlogram and the periodogram methods are based on a truncated estimate of the autocorrelation function, as the periodogram method implies an estimation of the autocorrelation function up to lag $\frac{1}{2} N_1$ and a certain window function. This shows the close relationship between the two methods.

More knowledge about the various estimates can be gained by investigating the underlying models for the time series.

RELATIONSHIP TO AR, MA, AND ARMA PROCESSES

The parametric processes associated with the above models may be generated by passing white noise through a recursive filter described by

$$x(nT) = \sum_{k=0}^{q} b_k u(nT - kT) - \sum_{i=1}^{p} a_i x(nT - iT) \quad (5)$$

where

- $x(nT)$ output of the filter
- $u(nT)$ white Gaussian process with unity variance
- $a_1, \cdots, a_p$ autoregressive (AR) parameters
- $b_0, \cdots, b_q$ moving average (MA) parameters.

The power spectrum of the autoregressive moving average (ARMA) process is

$$S(f) = \frac{T \left| \sum_{k=0}^{q} b_k \exp(-2\pi jkTf) \right|^2}{\left| 1 + \sum_{i=1}^{a} a_i \exp(-2\pi jiTf) \right|^2}. \quad (6)$$

Certain conclusions can be drawn about the processes.

For the AR process all b_k parameters except b_0 are zero. From (5) one can see that the present value of the time series is a linear sum of the past p values plus a sample from a random process. Alternatively, this can be stated as a linear prediction. The spectrum has only poles which is similar to the maximum entropy spectral estimate (3). It can be shown that least squares fitting of an AR model to any time series is equivalent to maximizing the entropy [9]. Therefore, in the following both the terms maximum entropy method and AR model fitting will be used to denote the same method.

By setting all the a_i parameters to zero, one obtains an MA process which has an all-zero spectrum, and whose present value is a linear sum of past and present values of an uncorrelated white noise process. The autocorrelation function is zero for lags greater than q. This is equivalent to the assumptions behind the periodogram and correlogram estimates, and shows that these spectral estimates contain only zeros.

Finally, the ARMA process has a pole-zero spectrum. By modeling an unknown time series with an ARMA model one would expect to need the fewest number of parameters as this is the most general linear model.

So far nothing has been said about how to obtain the various unknown parameters. The autoregressive model gives a linear equation that can be solved quite easily. By estimating the autocorrelation function up to lag p one can solve the Yule-Walker equation recursively [8]. This method requires about $p \cdot N$ multiplications (N is the length of the time series). When the time series is short it is better to use the Burg technique to estimate the parameters directly from the time series [1], [8]. This method requires about $5\, p \cdot N$ multiplications when solved recursively.

The main difference between the two methods lies in the criterion used for optimality. In the first method the criterion is computed from the forward prediction error power, but the computation involves assumptions that are not consistent with the maximum entropy principle. In the Burg method this is avoided, but it is instead necessary to calculate both the forward and the backward error power to ensure stability in the resulting filter.

A difficulty in implementing both algorithms is the choice of the number of parameters p. Various criteria have been suggested which include the final prediction error (FPE) criterion and the criterion autoregressive transfer (CAT) function [5]. We favor a simple test on the partial autocorrelation coefficient. This coefficient is the correlation between two samples when all linear regressions on the between-lying samples have been eliminated. Thus for an AR (p) process the partial autocorrelation is zero for orders greater than p. This coefficient can therefore be used to estimate the order. For a finite realization of an AR (p) process the variance of the partial autocorrelation for lags greater than p is [2]

$$\sigma^2 \approx \frac{1}{N} \tag{7}$$

where N is the number of samples from the process. A suggested test for the model order is to select the lag for which the partial autocorrelation drops below 2σ.

The ARMA model gives rise to a nonlinear equation [7]. It is therefore not as straightforward to use as the AR model. Besides there is the problem of choosing both p and q, the orders of the AR, and the MA terms.

MODEL FITTING TO WAVE SERIES

The wave series analyzed in this study were measured off the coast of Norway at two different places. They consist of 2048 samples taken at 2 Hz. The autoregressive spectral estimates have been computed with the Burg algorithm, to ensure that the algorithm used does not degrade the resolution of the spectral estimates. With the long series used here one would probably not have noticed much difference if the Yule-Walker algorithm had been used instead.

A Fortran subroutine for the Burg algorithm was taken from [8]. Some of the time series have also been prefiltered with one MA parameter before the computation of the AR parameters. The prefiltered series $x'(n)$ is obtained from the original series $x(n)$ by

$$x'(n) = x(n) - b_1 x'(n-1). \tag{8}$$

This prefiltering gives rise to a transient which has been removed by discarding the 200 first values of the series. The parameter b_1 used for prefiltering is substituted in (6) together with the computed set $b_0, a_1, \cdots, a_p$ to yield the spectral estimate. Thus the prefiltering introduces a zero in the estimate.

For each series the partial autocorrelation coefficient has been computed and tested against a 2σ threshold given by (7). This threshold has been plotted on all curves of the partial autocorrelation. The MA parameter b_1 used for prefiltering was chosen in such a way that the partial autocorrelation for the prefiltered series fell within this threshold for the smallest possible lag. We have also tested the FPE and the CAT criteria, but have found that they give a too low order when the spectra have two peaks close to one another.

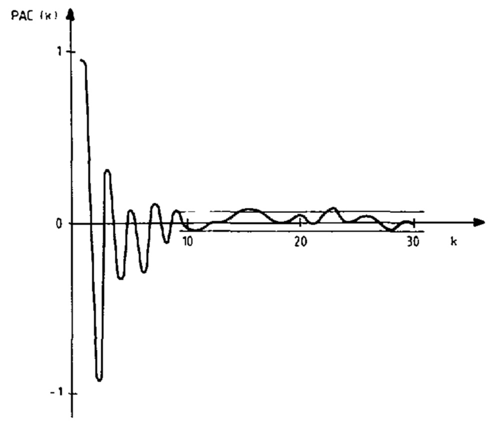

Fig. 2. Partial autocorrelation for series from Halten 77030621 (March 6 at 21 hours 1977).

The autoregressive spectral estimates have been compared with a smoothed periodogram spectral estimate with 16 degrees of freedom ($M = 8$, $N_1 = 256$ in (4)). This gives a relative standard deviation equal to $(8)^{-1/2} \approx 0.35$ on the smoothed periodograms. The standard deviation on the autoregressive spectral estimates is more difficult to compute and no attempt to do this is done here.

The first series has a spectrum with one peak. The partial autocorrelation suggests an order of about 15 (Fig. 2). For both the pure and the prefiltered AR model the peak moves to the low-frequency side with increasing model order (Figs. 3 and 4).

Compared to the periodogram (Fig. 5) both estimates give a smooth spectrum. The prefiltered AR(10) estimate seems to have lost the small peak at around 0.3 Hz, so it seems that both AR estimates need around 15 parameters to resolve all of the details.

The two main peaks in the periodogram in Fig. 5 are merged into one in the autoregressive estimates. However, as the relative standard deviation of the smoothed periodogram is 35 percent one cannot tell whether this is one or two peaks. Therefore, no attention is paid to this effect.

The next example is a spectrum with two peaks and $p = 30$ is suggested by the partial autocorrelation (Fig. 6). Compared to the previous example increasing the model order gives wiggles ($p = 40$, Fig. 7), that can best be explained by overfitting.

The AR(30) estimate is smooth and follows the periodogram very well (Fig. 8).

The last series is from Tromsøflaket and has one broad peak. The partial autocorrelation (Fig. 9) for this and other series from Tromsøflaket do not have as many oscillations as the Halten series, this is a difference which we have not been able to explain. For this last series, the partial autocorrelation suggests about 30 parameters in the pure AR model.

The pure AR estimate (Fig. 10) changes much with varying p. But none of the AR estimates are quite similar to the periodogram. The prefiltered AR estimate (Fig. 11) is more independent of p and gives a smooth estimate, but seems to have lost some details compared to the pure AR estimate and the periodogram (Fig. 12).

We conclude that the broad-peaked spectrum is one of the most difficult to model.

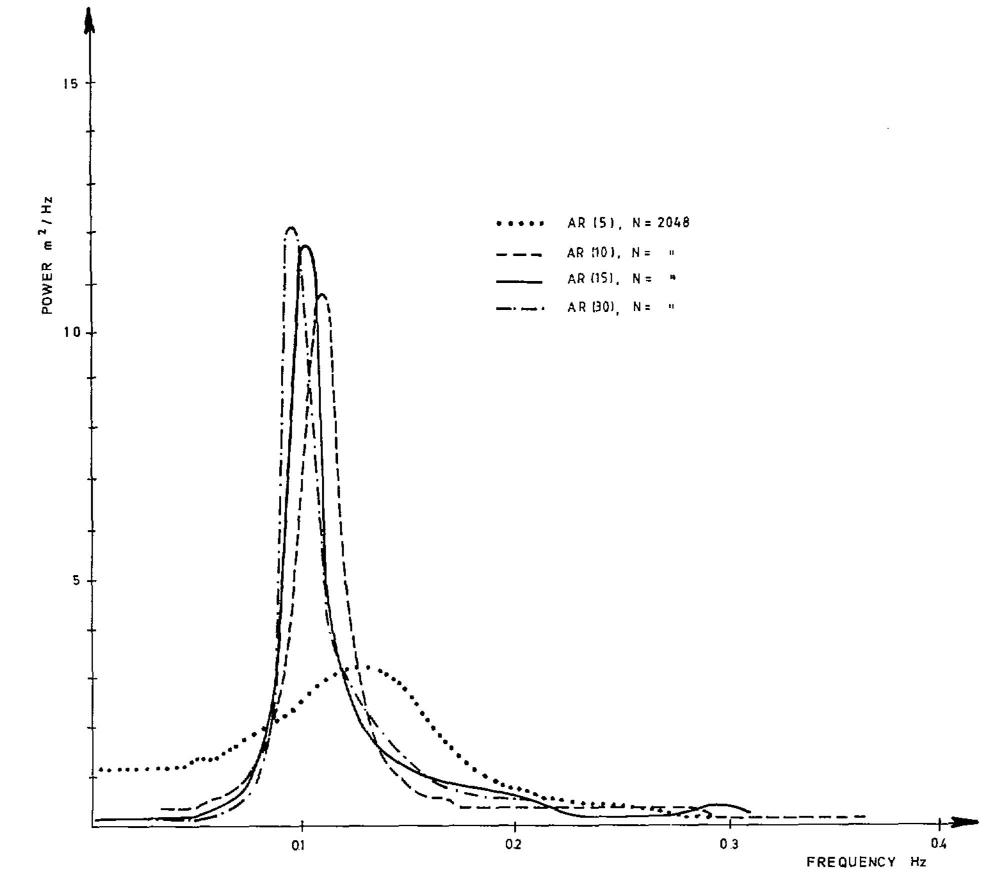

Fig. 3. AR-spectral estimates, Halten 77030621.

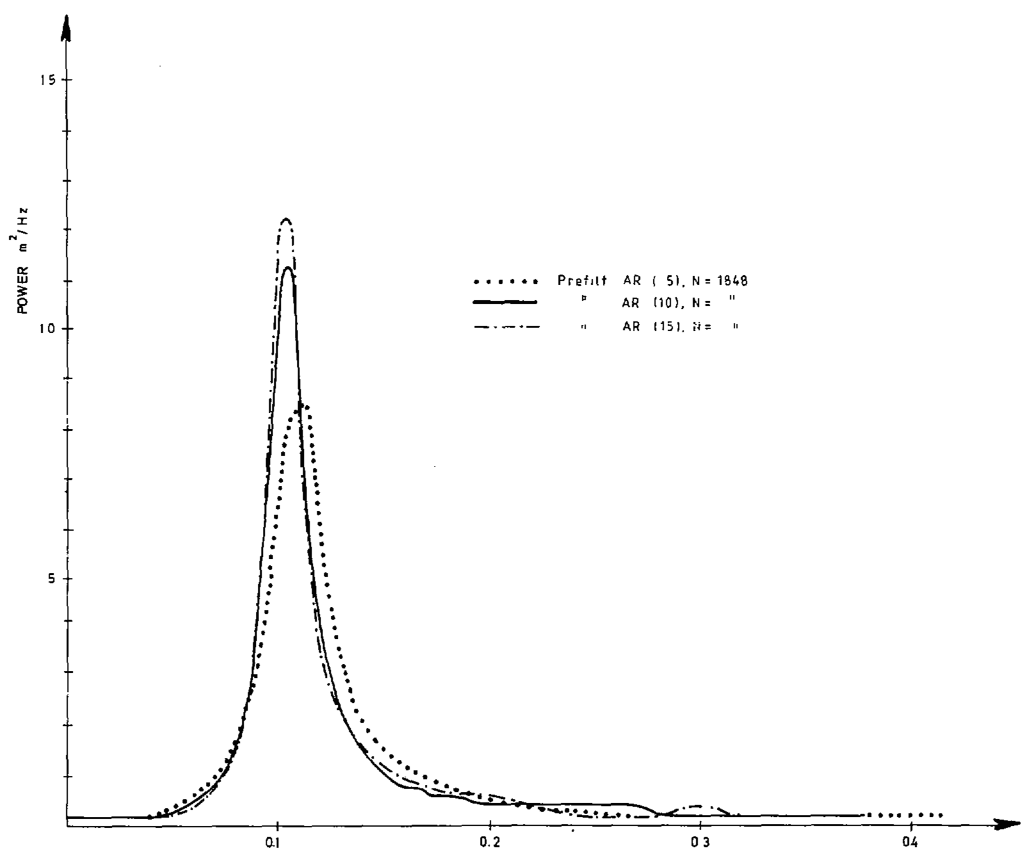

Fig. 4. AR-spectral estimates after prefiltering with $b_1 = -0.85$, Halten 77030621.

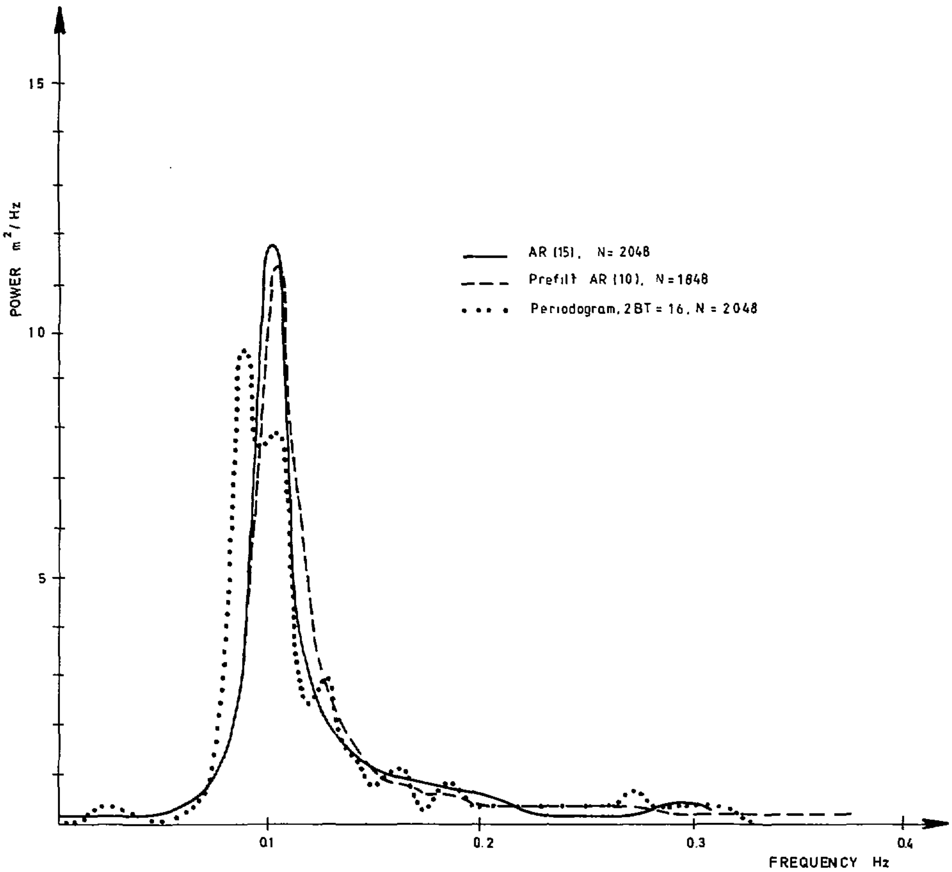

Fig. 5. AR, prefiltered AR, and periodogram spectral estimates, Halten 77030621.

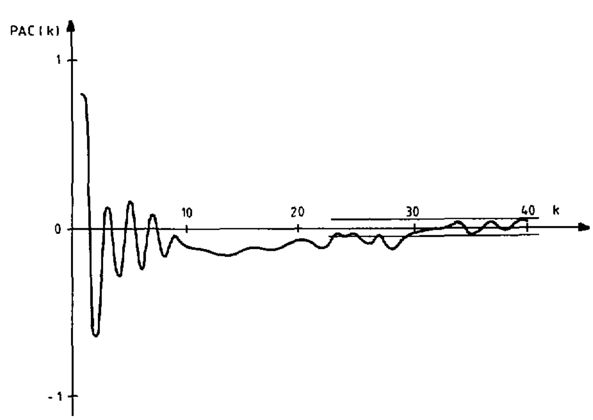

Fig. 6. Partial autocorrelation, Halten 77031803.

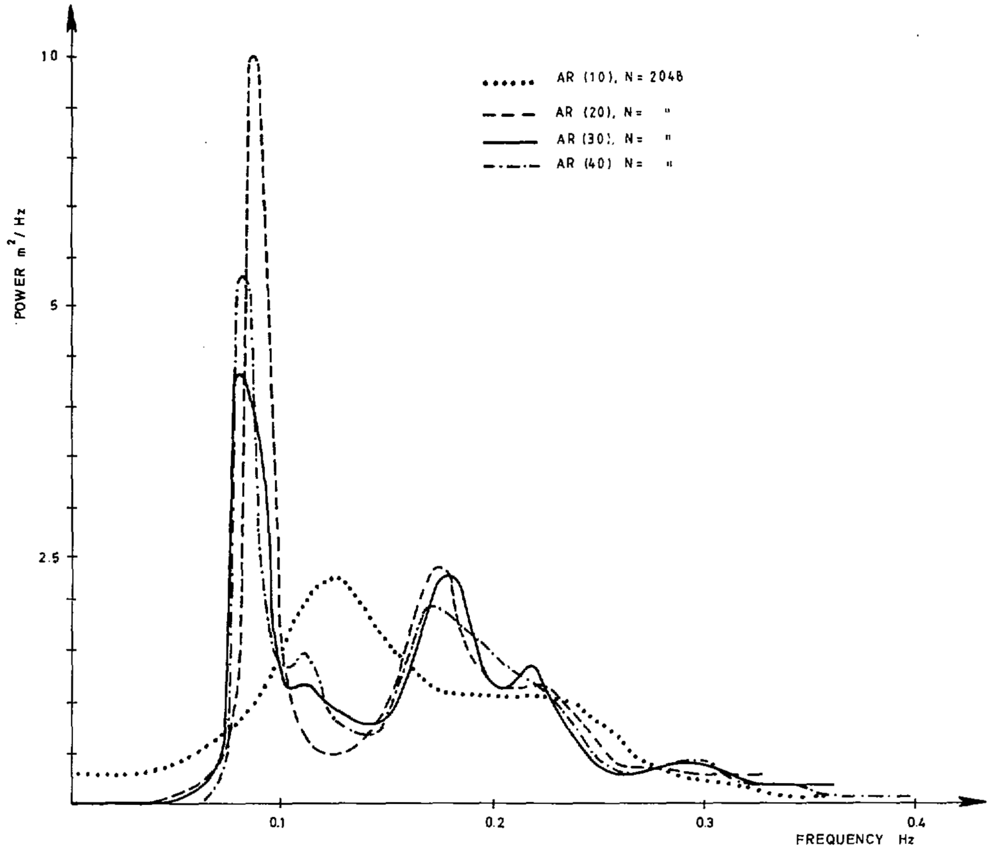

Fig. 7. AR-spectral estimates, Halten 77031803

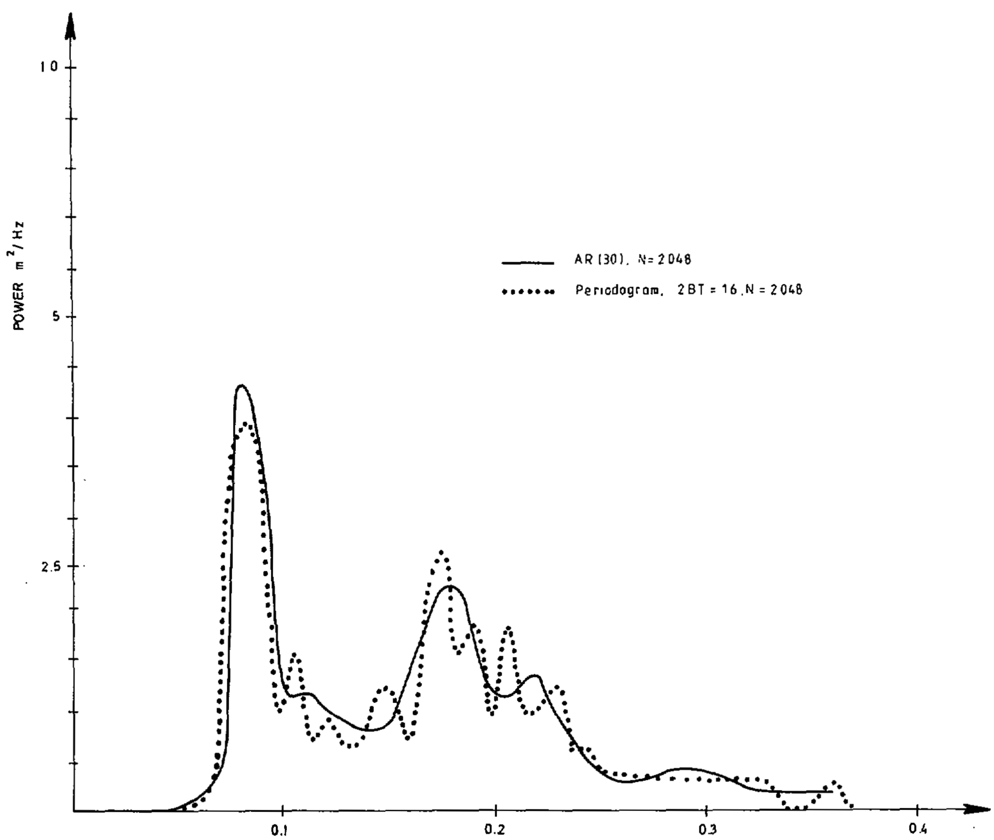

Fig. 8. AR and periodogram spectral estimates, Halten 77031803.

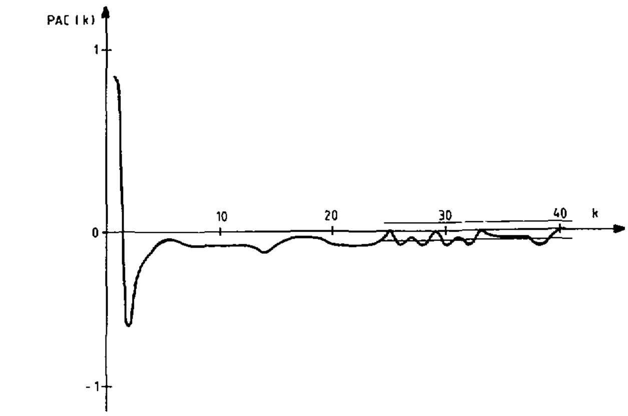

Fig. 9. Partial autocorrelation, Tromsøflaket 76120405.

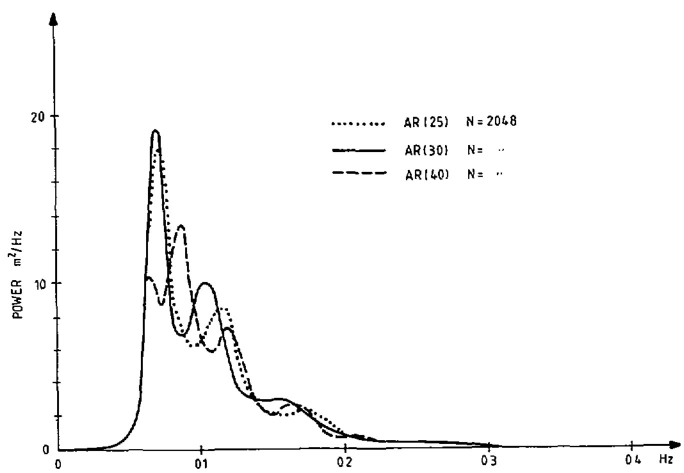

Fig. 10. AR-spectral estimates, Tromsøflaket 76120405.

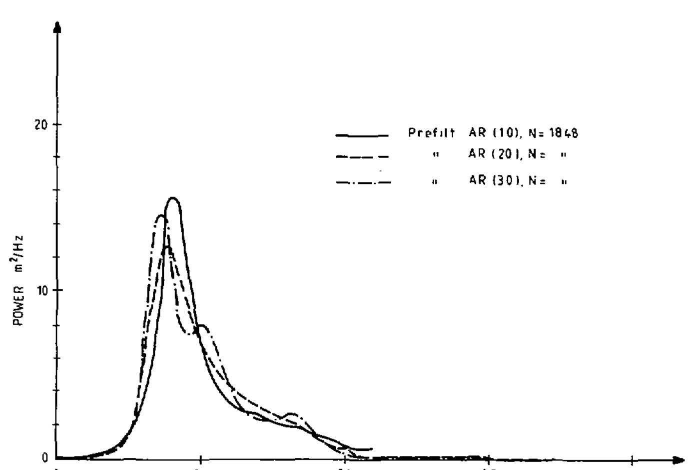

Fig. 11. AR-spectral estimates after prefiltering with $b_1 = -0.96$, Tromsøflaket 76120405.

In the previous descussion the new results have all been compared to the results from conventional spectral estimation. It should be remembered however that discrepancies could as well be due to the conventional estimates as to the parametric estimates.

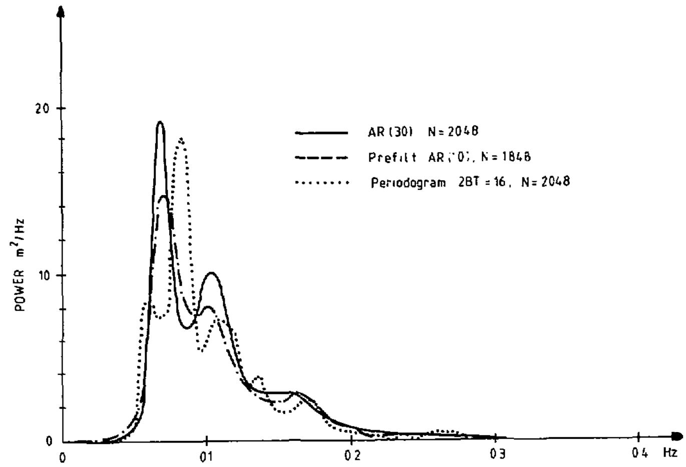

Fig. 12. AR, prefiltered AR, and periodogram spectral estimates Tromsøflaket 76120405.

TABLE I
MOMENTS CALCULATED FOR SERIES FROM TROMSØFLAKET

	M_0	M_1	M_2	M_4
Periodogram	0.845	0.098	0.0155	0.00139
AR(4)	0.845	0.095	0.0156	0.00143
AR(30)	0.848	0.098	0.0155	0.00140
AR(50)	0.868	0.097	0.0156	0.00139
Prefiltered AR(5)	0.802	0.095	0.0152	0.00141
Prefiltered AR(30)	0.803	0.094	0.0151	0.00139
Prefiltered AR(50)	0.810	0.094	0.0152	0.00139

CALCULATION OF MOMENTS

From the wave spectra estimates the moments up to order 4 are used to compute estimates of the significant wave height, the average wave period, and the number of crests during the data collection. The ability to give good estimates of the moments is therefore an important criterion for choosing between spectral estimates for ocean waves. The moments have been calculated by numerical integration over 89 spectral values

$$M_n = \frac{1}{128} \sum_{i=1}^{89} \left(\frac{i}{128}\right)^n S(i). \tag{9}$$

This is equivalent to integrating up to 0.7 Hz. The results have been compared to the moments calculated from the periodogram. Table I shows the moments calculated from the series from Tromsøflaket (Figs. 10-12).

It should be noticed how little the moments vary with a changing number of parameters. Even the AR(4) estimate which does not at all look like a good spectral estimate gives good estimates of the moments. This is because the pole model for the spectrum best models the spectral peaks and these peaks are the major contributors to the moments.

The discrepancy between the moments calculated from the pure and the prefiltered AR spectra is because the 200 first data samples are discared when the series is prefiltered. This gives a difference because the time series are not quite stationary.

REAL-TIME PROCESSING AND DATA REDUCTION

The algorithms for autoregressive model fitting have been tried out on a Texas Instruments 990/4 microcomputer programmed in Fortran. The purpose was to find the time needed on a 9900-series microprocessor. After conversion from the microcomputer to the microprocessor the needed time was about 100 s (10 percent of real time) for the Yule-Walker algorithm and about 500 s (50 percent of real time) for the Burg algorithm. This was with $p = 20$ parameters.

With the proper memories for the program and the time series, either analog or digital filtering to reduce the influence of noise and the effect of aliasing, an analog to digital converter for the input to the processor, and a transmitter for the digitally coded autoregressive parameters, a wave measuring and processing system could be built in the buoy. This way the needed storage and processing after transmission to the coast will be significantly reduced.

WAVE RECONSTRUCTION

After having obtained the autoregressive parameters a time series with the same statistical properties as the original series can be reconstructed. The regenerated waveform $x(nT)$ for an AR process is then given by

$$x(nT) = b_0 \cdot u(nT) - \sum_{i=1}^{P} a_i x(nT - iT) \tag{10}$$

where $u(nT)$ is an uncorrelated random sequence of numbers.

Such a reconstruction could be used for the generation of wave situations for simulations on models, as the sampling interval T and the gain factor b_0 can be chosen to fit the reduction scale.

CONCLUSION

The wave series are seen to fit an AR model of dimension 15 to 30 with good results. With prefiltering fewer parameters are needed to give a good, smooth estimate, but prefiltering does not reduce the number of parameters needed to resolve all details. Both models give a smoother estimate than the smoothed periodogram.

To compute the moments up to order 4 a model with few parameters can be used. Our examples show that 4 parameters are sufficient.

Another advantage with the AR algorithms is that they are simple enough to be implemented by a microprocessor and that they can produce wave spectra estimates in real time. Secondly, the spectrum is described with relatively few parameters which directly can be used for the reconstruction of waves with the same statistical properties. The few parameters needed also greatly simplify transmission and storage of wave series data.

So far only a few cases have been analyzed and it is therefore necessary to obtain more experience with the method. But the results so far seem to indicate that the autoregressive model is well suited for the analysis of wave series.

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REFERENCES

[1] N. Andersen, "On the calculation of filter coefficients for maximum entropy spectral analysis," *Geophysics*, vol. 39, pp. 69-72, Feb. 1974.

[2] G. E. Box and G. M. Jenkins, *Time Series Analysis, Forecasting and Control.* San Francisco, CA: Holden-Day, 1976.

[3] J. P. Burg, "Maximum entropy spectral analysis," presented at the 37th Annu. Int. Meet., Soc. Exploration Geophysics, Oklahoma City, OK, Oct. 31, 1967.

[4] J. A. Edward and M. M. Fitelson, "Notes on maximum entropy processing," *IEEE Trans. Inform. Theory*, vol. IT-19, pp. 232-234, Mar. 1973.

[5] M. Kaven and G. R. Cooper, "An empirical investigation of the properties of the autoregressive spectral estimator," *IEEE Trans. Inform. Theory*, vol. IT-22, pp. 313-323, May 1976.

[6] T. E. Landers and R. T. Lacoss, "Some geophysical applications of autoregressive spectral estimates," *IEEE Trans. Geosci. Electron.*, vol. GE-15, pp. 26-32, Jan. 1977.

[7] J. Makhoul, "Linear prediction: A tutorial review," *Proc. IEEE*, vol. 63, pp. 561-580, Apr. 1975.

[8] T. J. Ulrych and T. N. Bishop, "Maximum entropy spectral analysis and autoregressive decomposition," *Rev. Geophys. Space Phys.*, vol. 13, pp. 183-200, Feb. 1975.

[9] A. Van Den Bos, "Alternative interpretation of maximum entropy spectral analysis," *IEEE Trans. Inform. Theory*, vol. IT-17, pp. 493-494, July 1971.