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SPECTRAL MOMENT MATCHING - A RATIONALE FOR MAXIMUM ENTROPY ANALYSIS

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The formula for the spectral moments is expanded in a series consisting of autocorrelation terms. By using the autocorrelation extrapolation inherent in the maximum entropy method (MEM) for spectral analysis, it is found that good estimates of the moments can be found from a very low order autoregressive model. Examples are given from the analysis of ocean wave data and speech and it is shown that only 2-5 parameters are required to get good estimates of the spectral moments.

1. INTRODUCTION

The maximum entropy spectral analysis method has received considerable inteiest over the last years as a high resolution spectral estimator. The method is equivalent to fitting an autoregressive model or an all-pole model to the data and these terms are used to denote the same method. Here we will briefly review the rationales behind using this method instead of the periodogram based spectral analysis methods. In addition, a rationale, namely that of spectral moment matching, is presented.

The reasons for using the maximum entropy method can be grouped in two categories. The first category includes those applications where the maximum entropy principle is specifically needed. Typical are situations where one is interested in periodicities whose frequencies are lower than the inverse of the length of the time series. These periodicities can not be resolved with conventional methods since they assume either a periodic extension of the data or that the data is zero outside of the analysis interval. The maximum entropy method, however, can resolve them since there are no assumptions about the unavailable data built into the method. Applications are often found in geophysics, for instance analysis of natural seismic events, analysis of perturbations in the orbital data of planets and stars, and analysis of periodicities in sunspot data.

The second category of reasons for using the maximum entropy method is based on the similarity between the autoregressive model and the way the signal to be analysed is generated. This is the case in speech analysis. There is a one to one connection between an acoustic tube model consisting of discrete sections of varying cross-sectional area excited in one end, and an autoregressive model with as many parameters as there are sections. The same is true in seismic data processing, where a layered earth model corresponds to an autoregressive model.

In this paper we show that the spectral moments of a maximum entropy spectral estimate has some unique properties. The spectral moment estimate of order n of a one-sided, discrete N-point spectral density, S(f), is

$$\hat{m}_n = \frac{2}{N} \sum_{\ell=0}^{(N/2)-1} \hat{S}(\ell)\,(\ell/N\Delta t)^n \tag{1}$$

where Δt is the sampling period.

The moments give information about the distribution of power along the frequency axis. In the case of maximum entropy analysis the spectral density is expressed as an all-pole spectral estimate:

$$\hat{S}(f) = \frac{P_M \cdot \Delta t}{\left|1 + \sum_{n=1}^{M} a_n \exp(-j2\pi n f \Delta t)\right|^2} \tag{2}$$

where P_M is a power term and a_m for $m = 1,2,\ldots,M$ are a set of autoregressive coefficients. With such a spectral estimate the moments are almost invariant to the model order M. We call this the spectral moment matching property of the spectral estimator. This property is closely related to the maximum entropy extrapolation inherent in the method.

2. APPLICATIONS OF SPECTRAL MOMENTS

2.1 Ocean Waves

The distribution of the maximum of the ocean wave amplitude, a, is generally assumed to follow a Rayleigh distribution [1]. This is strictly only correct when the wave height has a narrow frequency spectrum. The r.m.s. width of the spectrum is often defined as

$$\varepsilon = 1 - \frac{m_2^2}{m_0 m_4} \tag{3}$$

The Rayleigh distribution is correct for $\varepsilon = 0$, in practical cases ε may be up to 0,5 (with $\varepsilon = 1$ being a white spectrum), but the distribution is still in good agreement with measured values.

An often used term in ocean wave analysis is the significant wave height, defined as the height which 1/3 of the maximas are above. For the Rayleigh distribution one will find that this is:

$$H_{1/3} = 4.0043\sqrt{2\pi m_0} \tag{4}$$

Another parameter estimated from the spectral moments is the mean zero-upcrossing period, its estimator is:

$$T_{02} = \sqrt{\frac{m_0}{m_2}} \tag{5}$$

The final parameter is the mean period between crests, whose estimator is:

$$T_{24} = \sqrt{\frac{m_2}{m_4}} \tag{6}$$

The set of spectral moments of order 0, 2, and 4 are therefore of vital importance for the estimation of characteristic parameters of an ocean wave time series.

2.2 Speech Analysis

In speech analysis the spectral moments have been used to extract formant frequencies [3]. The speech signal was preemphasized with 6 dB/octave and two estimates were formed:

$$F_{01} = \frac{m_1}{m_0} \tag{7}$$

$$F_{12} = \frac{m_2}{m_1}$$

They were used to separate the regions of the first and second formant frequencies. Except when the two formants, F_1 and F_2, are very close the following relations hold:

$$F_1 < F_{01} < F_2 < F_{12} \tag{9}$$

The procedure for formant estimation is to compute F_{01} and F_{12} and then estimate F_1:

$$\hat{F}_1 = \frac{m_1'}{m_0'} \tag{10}$$

where m_1' and m_0' are found by integrating (1) up to F_{01}. Likewise the second formant is estimated as

$$\hat{F}_2 = \frac{m_1''}{m_0''} \tag{11}$$

where m_1'' and m_0'' are found by integrating (1) from F_{01} to F_{12}. This gives a simple procedure for formant estimation, but with some inaccuracies (up to about ± 10 %, se [3]).

3. THEORY

The discrete time formula for the spectral moments (1) can be written as an infinite series by expanding the $(\ell/N\Delta t)^n$ term in a cosine series

$$\hat{m}_n = \frac{2}{N}\sum_{\ell=0}^{(N/2)-1} \hat{S}(\ell)\left\{\frac{c_0^n}{2} + \sum_{k=1}^{\infty} c_k^n \cos\left(2\pi\frac{k}{N}\cdot\ell\right)\right\} \tag{12}$$

The cosine expansion is symmetric around $\ell = 0$ and periodic with period N. Within the range of summation in (12), however, the expansion is equal to $(\ell/N\Delta t)^n$ as desired. The autocorrelation function can also be expressed as a cosine-series by using the discrete-time Fourier transform inverse of the spectral estimate:

$$\begin{aligned} R(k) &= \frac{1}{N}\sum_{\ell=-(N/2)}^{(N/2)-1} \hat{S}(\ell)\exp\left(j2\pi\frac{k}{N}\cdot\ell\right) \\ &= \frac{2}{N}\sum_{\ell=0}^{(N/2)-1} \hat{S}(\ell)\cos\left(2\pi\frac{k}{N}\cdot\ell\right) \end{aligned} \tag{13}$$

Combining (12) and (13) the spectral moments can be expressed in terms of the autocorrelation function via

$$\hat{m}_n = \frac{c_0^n}{2}R(0) + \sum_{k=1}^{\infty} c_k^n R(k) \tag{14}$$

The coefficients are

$$c_k^n = 2\int_{-0.5}^{0.5} \left|\frac{x}{\Delta t}\right|^n \cos(2\pi kx)\,dx \tag{15}$$

The solution to this integral is given in [2].

The coefficients decay as k^{-2} with increasing number of terms. The first terms in the series are therefore the most important.

In the maximum entropy method there is an inherent extrapolation of the autocorrelation function above lag M (the number of autoregressive parameters). This extrapolation is

$$R(M+k) = -\sum_{k=1}^{M} R(M+k-n)a_n, \quad k > 0 \tag{16}$$

With M + 1 values of the autocorrelation function estimated and the rest maximum-entropy extrapolated, the expression for the moments can be written with two terms:

$$\hat{m}_n = \left[\frac{c_0^n}{2}R(0) + \sum_{k=1}^{M} c_k^n R(k)\right] + \sum_{k=M+1}^{\infty} c_k^n R(k) \tag{17}$$

This expression for the spectral moments will be quite accurate even for small lags, M. The number of lags required will be dependent on how quickly the autocorrelation function decays.

4. EXAMPLES

4.1 Ocean Wave Spectrum

The first example is an ocean wave spectrum with two peaks, shown in Figure 1 for 5 and 30 autoregressive parameters. Thirty parameters are necessary to give good agreement with the averaged periodogram analysis method [4]. The sampling frequency is 2 Hz.

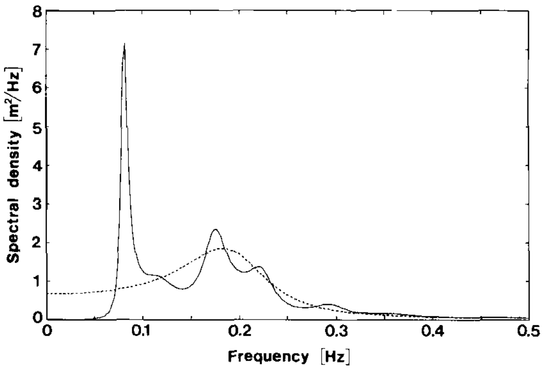

Figure 1. Spectral density of ocean wave data, 5 (broken line) and 30 (solid line) autoregressive parameters.

As an example the second-order spectral moment has been calculated by (17) with M, the autoregressive order, as abscissa (Figure 2). The extrapolation goes from M + 1 to 100. The circles in the plot (for M = 5 and M = 30) are the values obtained by integrating the spectra of Figure 1. ((1) with N = 1024.)

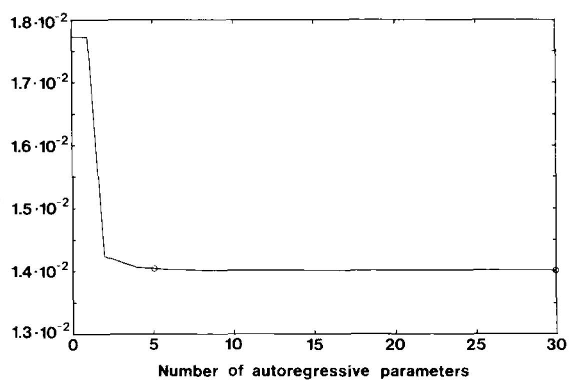

Figure 2. Second-order spectral moment versus increasing number of autoregressive parameters.

First one will notice that the spectral moment can be calculated from a model whose order is much lower than what is necessary to get a good spectral estimate. Second, the indirect calculation of moments via the extrapolated autocorrelation (17) is very accurate. The same convergence of the spectral moments holds true at least up to order 4 which is the maximum order tested [2]. The moments have been used to estimate the mean zero up-crossing period and the mean period between crests. The results are T_{02} = 4.816 seconds and T_{24} = 2.557 seconds for a 30. order model. With a 5. order model T_{02} decreases with only 0.08 % and T_{24} with 0.9 %.

4.2 Speech Spectrum

The second example is a 32 ms long segment of the vowel [a:]. The signal has been sampled at 8 kHz, preemphasized with one pole and weighted with a Hamming window prior to analysis. Figure 3 shows the spectrum.

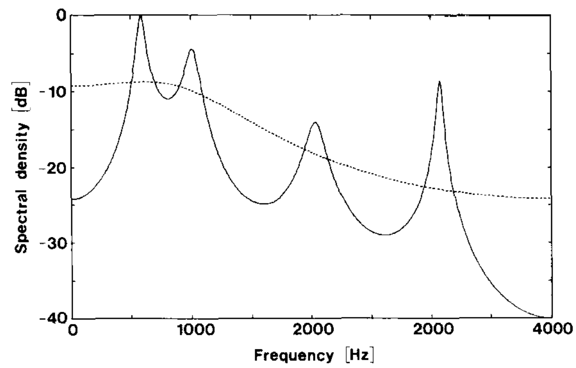

Figure 3. Spectral density of preemphasized speech. 2 (broken line) and 10 (solid line) autoregressive parameters.

From the spectral moments the estimators used for separating the regions of the first and second formants were found. With M = 10 autoregressive parameters F_{01} = 931 Hz which is between the first and second formants (F_1 = 570 Hz, F_2 = 1000 Hz). F_{21} was found to be 1345 Hz, thus separating the regions of the second and third formant (F_3 = 2050 Hz). With only 2 autoregressive parameters F_{01} = 873 Hz and F_{21} = 1443 Hz. It is remarkable that although no formant frequencies can be found from this spectral estimate the regions of the first and second formants can still be found.

5. CONCLUSION

We have shown that when estimates of the spectral moments are required the maximum entropy method will give these with a model dimension which is very small compared to the dimension needed to get a good spectral estimate. The moments can either be found by integrating the spectral estimates or indirectly via the extrapolated autocorrelation function. This might be a rationale for using this method since it requires little computation, and if the data needs to be transmitted only a small capacity is needed. The rationale is related to the first rationale mentioned in the introduction since it relies heavily on the maximum entropy principle.

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