Optimum FFT-Based Frequency Acquisition with Application to COSPAS-SARSAT

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In the case of a single sinusoid or multiple well-separated sinusoids, a coarse estimator consisting of a windowed Fourier transform followed by a fine estimator which is an interpolator is a good approximation to an optimal frequency acquisition and measurement algorithm. The design tradeoffs are described, and for the fine frequency estimator it is shown that a simple and good method is to fit a Gaussian function to the fast Fourier transform (FFT) peak and its two neighbors. This method is able to achieve a frequency standard-deviation and a frequency bias in the order of only a few percent of a bin. In the case of short-time stationarity a proper compromise between FFT length and number of averages has to be made. We show here that for moderate number of averages and for an adaptive threshold detector, only between 0.5 and 1 dB is lost when averaging is traded off for FFT length, as contrasted to the asymptotic result of 1.5 dB.

The COSPAS-SARSAT satellite system for emergency detection and localization is used to illustrate the concepts described. It is based on emergency locator transmitters and emergency position indicating radio beacons existing prior to satellite system. These beacons have spectral characteristics that are nonideal for Doppler positioning. The described algorithm is analyzed theoretically and good agreement is found with test results.

I. INTRODUCTION

Frequency acquisition and measurement is needed when near-sinusoidal signals are to be detected and the frequency found. Often such signals are man-made and may for instance be the carrier in a communications systems which is subject to unknown frequency shifts or the return from a moving target with small Doppler-spread in a narrowband radar system. Another example is the COSPAS-SARSAT system where the signals are due to one or several emergency beacons, subject to an unknown frequency shift caused by Doppler shift over a satellite link. In this application, the signal itself may be nonstationary in that the carrier is drifting.

We discuss frequency acquisition and measurement under two assumptions. The first is that the signals may be considered to be short-term stationary. The second is that if multiple signals are present, they can be assumed most of the time to be far enough separated to be resolvable without having to resort to high-resolution spectral analysis. In this case a Fourier-based method is a close approximation to a maximum likelihood (ML) frequency estimator and an optimum detector.

Frequency acquisition and measurement is an area that draws on two fields: detection theory [2], and sinusoidal parameter estimation [3, 4, 16]. According to [3] the algorithm consists of two steps after filtering and analog to digital conversion: 1) acquisition and coarse frequency estimation, and 2) fine frequency estimation.

The various blocks in the algorithm are discussed here and the COSPAS-SARSAT system is used as an illustrative example throughout. First an introduction is given to the COSPAS-SARSAT system and the signals encountered there. The ML frequency estimator achieves the maximum frequency accuracy, and its approximation is a Fourier transform preceded by a carefully selected window both in the single sinusoid and multiple, well-separated sinusoid cases. The Fourier transform may be realized by a fast Fourier transform (FFT). The same method is also an approximate matched filter, making it close-to-optimum with respect to detection at low signal-to-noise ratios (SNR). The frequency accuracy is discussed, a simple method for accurate fine frequency estimation is introduced, and the detection threshold is found.

Stationarity considerations leading to a tradeoff between coherent (FFT) and incoherent (averaging of FFT outputs) processing, are then discussed with the resulting change in detection thresholds. Finally a practical algorithm used in the COSPAS-SARSAT system is presented.

II. THE COSPAS-SARSAT SYSTEM

SARSAT is an acronym for search and rescue satellite-aided tracking and is a satellite-based system
established and operated by Canada, France, and the USA. It is operated jointly with the compatible COSPAS program of Russia. COSPAS-SARSAT has demonstrated that the detection and location of distress signals can be facilitated by global monitoring based on low-altitude spacecraft in near polar orbits. Complete coverage of the Earth, including the polar regions, can be achieved using simple emergency transmitters to signal a distress.

The system is based on emergency locator transmitters (ELTs) onboard aircraft, and marine emergency position indicating radio beacons (EPIRBs) at 121.5 and optionally at 243 MHz. When activated, their signals are received by the satellites and relayed down to a local user terminal. The estimated shape of the Doppler shift curve on the up-link over a maximum of about 15 min (Fig. 1) is used by the local user terminal in combination with precise estimates of the orbit of the satellite to find the location of the emergency. The position data is then sent from the local user terminal to a mission control center and then to the appropriate rescue coordination center which conducts the search and rescue operation.

Norway was the first country to join the COSPAS-SARSAT system outside of the space-segment providers. Early on an effort was therefore made to develop a national local user terminal [7]. This development took place in parallel with similar development in Canada and USA [8, 10, 12]. The frequency measurement algorithms together with the Doppler positioning algorithms described in [15] are key elements to the success of the local user terminal. The aim of the processing algorithms is to achieve an accuracy which is so high that it is essentially limited by the characteristics of the beacons. Since their specifications were determined prior to the establishment of the satellite system, the frequency stability of the beacons is not as high as would be desirable. Therefore a target accuracy of 20 km was the design goal of the system. Later a new system using 406 MHz beacons specifically designed for the COSPAS-SARSAT system was established. With this system, location accuracies in the order of one km and less is routinely achieved [15]. This is equivalent to a root-mean square error of about 1 Hz in the frequency measurement [11], when there are 10 measurements spaced 50 s apart. Since the location accuracy is proportional to relative frequency error and inverse proportional to the number of measurements, the same location error corresponds to a frequency accuracy of about:

$$\Delta f_{\text{rms}} = 1 \text{ Hz} \cdot \frac{121.5 \text{ MHz}}{406 \text{ MHz}} \cdot \frac{50 \text{ s}}{2 \text{ s}} = 7.5 \text{ Hz}$$ (1)

with measurements spaced 2 s apart in the 121.5 MHz band. This is achievable from the point-of-view of the frequency estimator, as is shown here. However the beacon signal itself is usually not stable enough to give this kind of accuracy.

The beacon signal is characterized by a modulation that is designed to be audible to a human ear, but not so well suited to automatic measurements. The signal consists of a characteristic downchirp that is repeated 2–4 times a second. The main signal parameters are given in Table I. As part of the development of detection algorithms, extensive characterization of beacon signals has been performed by measuring both short-term spectra and long-term stability [9]. The result is that beacon signals may roughly be divided into two groups, coherent and incoherent [14], depending on the spectral characteristics. A coherent 121.5 MHz beacon is defined as having at least 30% of its energy within ±30 Hz of the carrier frequency. The short-time spectrum of such a beacon is shown in Fig. 2. In the random-phase incoherent beacons the crystal oscillator is switched on and off by the pulse modulation causing phase frequency shifts between each modulation period. In most beacons the crystal oscillator is allowed to run continuously and the pulse modulation only turns the output signal on and off. In a well-designed circuit this results in a coherent signal, while in the frequency-pulled incoherent beacons there is still some interaction between the output stage and the oscillator. Short-time spectra of incoherent beacons are shown in Figs. 3 and 4. A model based on sinusoidal carrier with slowly varying center frequency is used here as a first approximation to the beacon signal, and we show later that even the incoherent beacons have stable carriers over a short time-span.

In the local user terminal the satellite downlink signal is received, demodulated, down-mixed to

![Fig. 1. Doppler shift as COSPAS or SARSAT satellite travels over beacon on-board a ship. Zero Doppler point occurs at the time of closest approach (TCA).](image)

**Table I**

<table>
<thead>
<tr>
<th>Main Characteristics of Beacon Signal</th>
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<tbody>
<tr>
<td>Carrier frequency</td>
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<tr>
<td>Frequency tolerance</td>
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<td>Frequency drift</td>
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<td>Modulation type</td>
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<td>Percentage modulation</td>
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<td>Modulation frequency</td>
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<tr>
<td>Sweep period</td>
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**TABLE I**

<table>
<thead>
<tr>
<th>Carrier frequency</th>
<th>121.5 MHz (optional 243 MHz)</th>
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<tbody>
<tr>
<td>Frequency tolerance</td>
<td>±50 ppm</td>
</tr>
<tr>
<td>Frequency drift</td>
<td>up to ±10 Hz/min</td>
</tr>
<tr>
<td>Modulation type</td>
<td>pulse, duty cycle 33% and 55%</td>
</tr>
<tr>
<td>Percentage modulation</td>
<td>&gt; 85%</td>
</tr>
<tr>
<td>Modulation frequency</td>
<td>downward swept over 700 Hz</td>
</tr>
<tr>
<td>Sweep period</td>
<td>0.25–0.5 s</td>
</tr>
</tbody>
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baseband, and antialias filtered before analog to digital conversion. The data samples are then processed by the frequency acquisition and measurement system. The resulting time-tagged frequency detections are then processed by the Doppler positioning algorithm. The sampling frequency is determined by the relayed band which is 25 kHz wide. This bandwidth is computed from the maximum Doppler shift in the 121.5 MHz band which is ±2.9 kHz and the frequency tolerance of ±6 kHz. All beacon signals, therefore, lie within a band of approximately 18 kHz centered on 121.5 MHz. Real sampling at 51.2 kHz is therefore used.

III. SINGLE DFT AS AN APPROXIMATE ML ESTIMATOR

Cramer-Rao Bound, ML Estimation, and Optimum Detection

The lower limit on frequency error for any unbiased estimator is given by the Cramer-Rao bound. Assuming a single real sinusoidal signal embedded in
white, Gaussian noise, with unknown frequency, unknown amplitude, and unknown phase, it is derived by Rife and Boorstyn [3] to be

\[
\sigma_f = \frac{\sqrt{12}}{2\pi T \sqrt{N(N^2 - 1) \text{ SNR}}}
\]

(2)

where \( T \) is the sampling period, \( N \) is the number of samples available, and \( \text{SNR} \) is signal-to-noise ratio. In order to achieve this limit, an ML estimator should be used. It can be shown that the ML estimator in this case is [3]:

\[
X_{\text{ML}} = \max_f \left| \frac{1}{N} \sum_{n=0}^{N-1} x_n \exp(-j2\pi f nT) \right|^2
\]

(3)

where \( x_n, n = 0, \ldots, N - 1 \) are the available input samples. When the maximizing frequency is found, the maximum is also the estimate of the power of the sinusoidal. The Cramer-Rao bound describes well the performance of the ML estimator for signal-to-noise-ratios down to a certain threshold. Below this threshold the maximization becomes unreliable in that the peak occurs at random frequencies.

The ML estimator is equivalent to an optimum detector [2], and also has a close relationship to the discrete Fourier transform (DFT). These analogies are now exploited in order to find the threshold SNR and to find a practical algorithm. First consider a detection problem with an unknown, constant amplitude and random uniformly distributed phase. The receiver operating characteristic for the matched filter detector of [2, Ch. 7] gives the threshold levels. They are plotted with the solid line in Fig. 5 for varying probability of false alarm and for detection probabilities of 50% and 95%.

The Fourier transform gives values of the ML estimator computed over an evenly spread set of frequency bins given by the number of samples in the transform. Thus an approximate ML estimation or matched filtering can be done in two steps, where the first is a coarse acquisition based on squared DFT (periodogram) outputs. The second step is a fine estimation to be described later. When the FFT algorithm is used for the first step, the number of transform samples is in general \( rN \), where \( r \) is one or an even number. The input samples for the transform are \( x_n, n = 0, \ldots, N - 1 \) and 0 for \( n = N, \ldots, rN - 1 \) where the factor \( r \) indicates the amount of zero-padding in the transform.
The effect of not being able to maximize the likelihood function exactly has been described by Harris [5]. It manifests itself in the form of a loss of SNR that results in a decreased ability to detect. Assuming no zero-padding \((r = 1)\), the maximum loss occurs when the unknown frequency falls exactly halfway between two frequency bins. It is denoted scalloping loss and is shown in a dashed line in Fig. 6 for the case of a DFT taken directly on the data samples without any windowing (rectangular weighting). The scalloping loss is in this case 3.92 dB. In order to reduce this loss an \(N\)-point smooth window function \(w_n\) is used to multiply the data samples, so that \(x_n \cdot w_n\) are the input values for the transform. Thus in Fig. 6 a typical frequency response of a window function is also plotted (solid line). The plotted response is that of the Hamming window defined by

\[ w_n = 0.54 - 0.46 \cdot \cos(2\pi n/N), \quad n = 0, \ldots, N - 1. \]  

This window has a scalloping loss of 1.78 dB and therefore gives a more uniform sensitivity than rectangular windowing. The drawback is that the window reduces SNR even when the unknown frequency falls exactly in the center of a bin. This processing loss (PL) is 1.34 dB. It is the inverse of the window's Equivalent Noise Bandwidth (ENBW) which is given by the ratio of the incoherent gain and the coherent gain:

\[ \text{ENBW} = \frac{1}{\text{PL}} = \frac{1}{N} \frac{\sum_{n=0}^{N-1} w_n^2}{\left(\sum_{n=0}^{N-1} w_n^2\right)^2.} \]  

The figure of merit that describes the maximum loss in detection threshold is the worst case processing loss (WCPL) which is the sum of the scalloping loss and the processing loss. It should be subtracted from the threshold values given in Fig. 5 (solid line). For the Hamming window it is 3.10 dB which ranks it among the better windows. None of the windows listed in [5] have WCPL better than 3 dB. Thus the Hamming window gives a loss of 1.34 dB compared with no windowing for frequencies in the center of a bin, and an improvement of 3.92 - 3.10 = 0.82 dB for frequencies halfway between bins.

These figures can however be improved by using zero-padding in the transform. For instance in the case of zero-padding with \(N\) zeroes, \((r = 2)\), a new response centered at bin = 0.5 will be filled in between the responses at bin = 0 and bin = 1 in Fig. 6. The resulting WCPL is about 0.9 dB for a rectangular window and about 1.8 dB for Hamming windowing. The scalloping loss for the Hamming window is in this case reduced to only 0.47 dB.

**Multiple-Tone Estimation**

When there are multiple sinusoids present, the Cramer-Rao bound and the ML estimator are given by complicated expressions that in general are hard to analyze analytically [4, 16]. In general one has to use high-resolution spectral estimation methods in order to resolve the sinusoids. This is beyond the topic of this paper. An important case is however when the frequency difference between the sinusoids is greater than a critical frequency separation of \(2/NT\). In that case Rife and Boorstyn [4] have shown that the Cramer-Rao bounds for each of the tones approach that of a single tone. They also show that the FFT-based coarse estimator may be used as an approximation to the ML estimators for frequencies separated by more than \(4/NT\) or four bins.

It is a common misconception to confuse frequency accuracy and frequency resolution when high-resolution and classical FFT-based methods are compared. The \(N\)-point FFT for instance has a frequency resolution in the order of \(1/NT\) which is worse than many of the high-resolution methods. The frequency accuracy for single and multiple, well-separated tones approaches that of the Cramer-Rao bound for a well-designed fine-estimation stage following the FFT outputs. Most high resolution methods perform worse than that, and all of them have a higher signal to noise threshold (at least 4 dB higher according to examples in [16]).

In the case that the acquisition system is designed for multiple, separated, sinusoids, then not only the mainlobe characteristics of the window are important, but also the leakage between the sinusoids or the sidelobe properties. Both the maximum sidelobe level and the fall-off rate are important as they determine the bias caused by nearby interference and distant interference, respectively. The ultimate choice of window function should be dictated by the number of sinusoids expected, and their relative level differences. In the case of the COSPAS-SARSAT system, the processor must handle input signals with \(S/N_0\) in the range from 15 dB-Hz to at least 35 dB-Hz. The
processor must therefore have a dynamic range of at least 20 dB. The Hamming window is a simple window that gives a dynamic range of around 40 dB, and a close-to-minimum worst case PL. Its parameters are given in Table II (from [5]).

Fine Frequency Estimation

The FFT-based coarse frequency estimator has a bin-distance of $1/NT$. Every estimated frequency value is therefore truncated to the nearest bin, and this will increase the frequency uncertainty unless compensated for. If the tone frequency is uniformly distributed in $\pm$ half a bin, the resulting standard-deviation of the frequency estimator will be

$$\sigma_{f,\text{coarse}} = \frac{1}{2\sqrt{3}}1/NT. \quad (6)$$

In the worst case when the tone is located exactly halfway between two bins, the estimated value may jump between the two bins with equal probability. The resulting standard-deviation is in this case as high as half a bin. In Fig. 7 the Cramer-Rao bound from (2) has been plotted for the COSPAS-SARSAT case with a sample size of $N = 1024$. The SNR or carrier-to-noise ratio (dB) is given by the carrier-to-noise density (dB-Hz) and the noise bandwidth in each bin assuming that the carrier frequency is exactly located in a bin center:

$$C/N = C/No - 10 \cdot \log BW$$

$$= C/No - 10 \cdot \log(f_c \cdot ENBW/N) \quad (7)$$

and where $f_c = 51200$ Hz is the sampling frequency. This value is substituted for SNR in (2). The coarse rms frequency error of 14.4 Hz is also plotted in Fig. 7 for sake of comparison. This shows that there is a potential for at least an order of magnitude improvement in frequency accuracy if the effect of bin truncation can be overcome. There is therefore a need for a fine frequency estimator. Although this stage cannot make up for the loss in detection threshold given by the worst case PL, it improves frequency accuracy. The best solution is to implement this stage as a local maximization of the ML estimator (3) over a range of frequencies given by the maximum value from the FFT bank and its nearest neighbors. This gives a near-optimum solution, but a rather time-consuming one implementation-wise.

Rife and Boorstyn describe a simplified method based on fitting a linear function between the peak and the largest neighbor of the peak and with coefficients given by the window. Another alternative is to use a parabolic function forced to pass through the peak in bin $k$ and its two nearest neighbors. The bias of this method can be computed by using the noise-free response shown in Fig. 6 as input. The resulting bias is shown in Fig. 8 (solid line). The maximum bias is about 13% of a bin, depending on the window function used. The bias is a result of the mismatch between the response shown in Fig. 6 and the parabolic shape. A much better fit is achieved when a Gaussian function is used instead. It can be shown that a Gaussian fit is equivalent to fitting a parabolic function to the logarithm of the data. Thus (8) should be used with logarithmic input. The resulting bias is also shown in Fig. 8 and is reduced to about 2% of a bin.

Results from a simulation using the Gaussian fine frequency estimator is shown in Fig. 7 together with the Cramer-Rao bound. One should notice that the fine frequency estimator is useful down to accuracies in the order of a few percent of a bin. The discrepancy between the simulated values is due to the windowing and the fine frequency estimator.

HOLM: OPTIMUM FFT-BASED FREQUENCY ACQUISITION WITH APPLICATION TO COSPAS-SARSAT

\[469\]
Fig. 8. Bias in fine frequency estimator based on parabolic function (solid line) and Gaussian function (dashed line) with Hamming-windowed coarse frequency estimator.

Adaptive Threshold Calculation

In many cases one does not have a good estimate of the background level, or it may deviate from the white noise assumption so that it is not uniform over the output frequency range of the FFT. In any case it is difficult to apply the theoretically computed detection thresholds. One intuitively appealing method to overcome this problem is to estimate the background level adaptively, for instance by applying a symmetrical window around each frequency bin. For the COSPAS-SARSAT system a rectangular window which is about as wide as a beacon signal with modulation sidebands gives suitable smoothing. The threshold can be computed in the logarithmic domain making it insensitive to a few large values in the window, but still following the general trend of the frequency spectrum.

Another alternative is to assume that there is only a limited number of signals available at any instance of time, and regard the $K$ maximum values as detections. This method was also used by El-Naga and Carter [13]. If in addition all close neighbors that are passed as detections are merged into a single detection given by the frequency of the maximum of the cluster, a constant number of false detections is a more appropriate description of the detection process than the usual constant false alarm rate model. It is difficult now to proceed with the analysis unless a specific example is assumed. In the COSPAS-SARSAT detector the probability for false detection has been set on the assumption of four false detections per pass of the filter bank regardless of number of bins in the DFT:

$$P_F = \frac{N_{\text{false}}}{N/2} = \frac{8}{N}. \quad (9)$$

The resulting detection thresholds for a single DFT-processor in the 121.5 MHz band with a Hamming-window and $f_s = 51200$ Hz are given in Fig. 5 (dashed line). These results are valid for a pure sinusoid in the middle of a bin. In the worst case the scalloping loss has to be added.

IV. AVERAGED DFTS AND STATIONARITY CONSIDERATIONS

According to theory presented so far, frequency accuracy can be increased to any desired value by increasing the number of processed samples and the lengths of the DFT. The same applies to the detection threshold. There are two reasons why in practice this cannot usually be done. The first is simply due to hardware restrictions that may set a limit on the maximum size of the DFT. The second reason is that the signals may have to be regarded as nonstationary when the length of the input data exceeds a certain value. In any case, one will have to resort to a compromise between coherent processing achieved within a single DFT, and incoherent processing by averaging DFT outputs. Clearly this is not optimum since the ML structure of (3) prescribes coherent processing only. In this section an analysis of the criterion for trading incoherent and coherent processing based on stationarity is discussed, and the resulting degradation in detection threshold is found.

According to the matched filter theory of [2], the individual frequency bin envelope data from each of the $M$ DFTs should be added according to

$$Q(k) = \sum_{m=1}^{M} \ln(k_m q_m(k) \text{SNR})$$

$$\approx \begin{cases} \text{SNR}^2 \sum_{m=1}^{M} q_m^2(k), & \text{for small SNR} \\ 2 \text{SNR} \sum_{m=1}^{M} q_m(k), & \text{for large SNR} \end{cases} \quad (10)$$

where $q_m(k)$ is the envelope from DFT $m$ in bin $k$ and $k$ is the modified Bessel function of zero order. The quadratic approximation is the simpler one to analyze, but the linear approximation gives better performance by up to 0.2 dB for small number of averages [2]. Thus the differences are minor. In a fixed-point implementation, the linear sum is preferable since quadratic values require twice as many bits for representation.

The envelope is computed from the real and the imaginary parts:

$$q = \sqrt{(q_r)^2 + (q_i)^2} = q' \cdot \sqrt{1 + (q'/q_r)^2}$$

$$= q' \cdot \sqrt{1 + (q'/q'_r)^2} \quad (11)$$

In order to further simplify a fixed point implementation the square root can be expanded to...
first or zeroth order:

\[ q \approx \begin{cases} \frac{q' + q''}{2q'} \approx q', & q' \geq q'' \\ \frac{q' + q''}{2q'} \approx q'', & q' < q'' \end{cases} \] (12)

In practice the simplified method of accumulating the maximum of the real and imaginary parts gives negligible degradation compared with accumulation of the envelope or squared envelope.

Lack of stationarity may be modeled as a slow drift of the carrier specified by the drift-rate \( D \) in Hz/s. There is a tradeoff that results in an optimum DFT length for each specified signal given by the stationarity. The minimum processing bandwidth may be computed from the time \( T \) that it takes to sample data and the drift-rate, by specifying that the drift shall be maximum one filter bank bandwidth, \( BW \). It is defined as the noise bandwidth and is given by the equivalent noise bandwidth of the window and the time to sample data of a single DFT:

\[ BW = \frac{ENBW}{TDFT} = T \cdot |D|. \] (13)

When the filter bank is based on averaging \( M \) FFTs with an overlap of \( OL \) (0 < \( OL \) < 1.0), the measurement time is

\[ T = [OL + (1 - OL) \cdot M] \cdot TDFT \]

\[ = \frac{[OL + (1 - OL) \cdot M] \cdot ENBW}{BW}. \] (14)

The minimum bandwidth is obtained by combining the two equations:

\[ BW = \left( ENBW \cdot [OL + (1 - OL) \cdot M] \cdot |D| \right)^{1/2}. \] (15)

Finally, the DFT length is

\[ N = f_s \cdot TDFT = f_s \cdot ENBW / BW \] (16)

where \( f_s \) is the sampling frequency.

The bandwidth \( BW \) and the DFT length \( N \) are plotted for the COSPAS-SARSAT system in Fig. 9 with the drift rate \( D \) as parameter. In this figure the Hamming window is assumed with an overlap of 50%. The minimum value used for the drift rate is 25 Hz/s which is approximately the maximum Doppler drift in the 121.5 MHz band. The maximum drift rate for an incoherent beacon has been found from measurements to be 5000 Hz/s. In addition, a typical measurement time which is larger than several periods of a beacon, namely \( T = 900 \) ms, is shown.

For coherent beacons, where the maximum Doppler drift is the main contribution to lack of stationarity, averaging of up to about 300 1k DFTs or about ten 8k DFTs gives maximum sensitivity. This is roughly equivalent to the 163 1k DFTs found in a similar analysis in [12] since that analysis did not take into account overlap. A typical result is shown in Fig. 10 where the 1 s average of the 1k short-time spectra for the coherent beacon of Fig. 2 is shown. The carrier is clearly enhanced by the averaging.

For optimum detection of the portions of the incoherent beacon signals with maximum drift rate, single 1k DFTs are best suited. An investigation into beacon signal characteristics has however shown that most incoherent beacons have stable carriers, but that a smaller portion of the total emitted power is used for the carrier than for the coherent beacons. Figs. 11 and 12 illustrate this by showing that even incoherent beacons such as those of Figs. 3 and 4 have stable portions which are enhanced by the averaging the squared magnitude output of 100 1k DFTs. This is seen by the carriers that appear even for the incoherent beacons. An algorithm based on averaging of overlapping DFTs called the averaged periodogram method has therefore been developed. This method has a close relationship to a similar method described in [12]. The algorithm is suited for detection and accurate frequency measurement of all coherent and most incoherent beacons.

Detection Thresholds for Averaged DFTs

In the case of averaged DFTs, the detection thresholds presented in Fig. 5 are modified. Based on asymptotic results, it is well known that doubling the DFT length lowers the threshold by 3 dB, while doubling the number of averages lowers the threshold by only 1.5 dB. Given a constant amount of data, the effect of doubling the number of averages is the difference, i.e., 1.5 dB increase in detection threshold. There are however significant deviations from this rule-of-thumb when there is a small number of averages, i.e., less than a few hundred, and when the adaptive threshold model (constant number of false alarm rate) of (9) is assumed. When these cases are combined, the penalty for trading incoherent processing for coherent is not as severe as the asymptotic result indicates.

First the effect of overlapping transforms must be considered. Assuming that \( M \) DFTs are averaged, one can find the number of independent averages or
effective number of averages, \( M_e \). This value is what counts for instance in lowering the detection threshold. It is based on the overlap correlation of the window [5]:

\[
c_i(OL) = \frac{\sum_{n=0}^{N-1} w(n)w(n + j[1 - OL]N)}{\sum_{n=0}^{N-1} w^2(n)}
\]

(17)

where the window is assumed to be zero outside the range 0 to \( N - 1 \). For \( OL = 50\% \) overlap the only non-zero value (\( j = 1 \)) is listed in Table II. The effective number of averages in that case is [1]:

\[
M_e = M \left[ 1 + 2 \sum_{j=1}^{M-1} \frac{M-j}{M} c_i^2(OL) \right].
\]

(18)

With 50\% overlap the equivalent number of averages for the Hamming window is \( M_e \approx 0.9 \) \( M \). With 75\% overlap the Hamming-window gives \( M_e \approx 0.47 \) \( M \).

As an example consider the COSPAS-SARSAT system with 1 s of data sampled at 51200 Hz. No overlap gives \( M = M_e = 50 \) k DFTs, 50\% overlap gives \( M = 99 \) DFTs and \( M_e = 89 \), and 75\% overlap gives \( M = 197 \) DFTs and \( M_e = 93 \). There is a considerable gain by using 50\% overlap, but the gain from increasing to 75\% is not worth the doubling of the processing capacity that is necessary. An overlap of 50\% is therefore recommended.

The decrease in detection threshold due to variation in false alarm probability shown in Fig. 5 (solid line) varies from about 0.4 dB \( (P_d = 0.5) \) to 0.6 dB \( (P_d = 0.95) \) when the DFT-size is divided by two. This adds to the gain from averaging which varies...
The net result of trading incoherent processing for coherent processing given a constant amount of data is a decrease in detection threshold of minimum $(3 - 1.9 - 0.6) = 0.5$ dB and maximum $(3 - 1.6 - 0.4) = 1$ dB per doubling of the number of averages. Thus the averaging is more efficient than the asymptotic result of 1.5 dB implies. This makes it less important to choose long DFT lengths and makes the performance loss with nonstationary signals less than one would have thought.

The resulting detection thresholds for processing of 1 s of data by averaged DFTs of different sizes in the 121.5 MHz band with a Hamming window and $f_s = 51200$ Hz are given in Fig. 13. A detection probability down to 0.5 is assumed since 50% of the points of a Doppler curve are adequate for positioning. The beacon signals have a Doppler shift between -5 Hz/s and -26 Hz/s. This implies that over the processing interval, the signal will shift its position within the 50 Hz bin width. In the worst case the maximum scalloping loss of 1.78 dB will be experienced, giving a worst case detection probability of 14.5 dB-Hz with 1k DFTs.

V. AVERAGED PERIODOGRAM ALGORITHM FOR COSPAS-SARSAT

The implemented algorithm is based on a $N = 1024$ point Hamming-weighted DFT filterbank implemented by an FFT algorithm in an array processor. In the 121.5 MHz band, the bin distance is 50 Hz, while the noise bandwidth is 68 Hz. Based on the stationarity considerations in the previous section, and a desire to average over several beacon periods, 1 s has been chosen as the processing period. The data of each band is processed with a 50% duty cycle. The first second the 121.5 MHz band is processed, and the next second the 243 MHz band is processed and so on, giving a new set of detections every 2 s in each band. To give time for threshold calculation and transfer of results from the array processor each second, in order of 900 ms of data can be processed in each band. A 50% overlap between windows is used for processing of the 121.5 MHz band. This gives 90 1024 point DFTs that span 910 ms which corresponds to about 0.2 dB higher thresholds than predicted for 1 s of data in Fig. 13.

After the averaging, an adaptive threshold is computed in order to make the detector less sensitive to variations in level over the frequency band. This level variation is caused by nonflat frequency characteristics in the satellite transponders. In addition the adaptive threshold improves the performance when strong and weak signals are present simultaneously. The detection process is done in two passes. In
the first pass all peaks are found. These values are then sent to the second pass which is a cluster peak detector. It reduces the number of detections by retaining only the peak when a cluster of detections is found in the first pass. The peak detector has a characteristic bandwidth which is the minimum frequency distance allowed between two detections. This further reduces the number of false or duplicate detections. This is important in order to simplify the Doppler curve sorting and Doppler positioning, since the number of candidates for curves and positions is reduced.

Test tapes containing calibrated coherent and incoherent beacon signals with simulated Doppler shifts have been processed in the ground station. A system test on coherent beacons using an early version of the Doppler positioning software of [15], resulted in beacons at signal-to-noise densities of 15, 20, and 25 dB-Hz being detected and localized to 8.4, 1.1, and 6.2 km error, respectively. Beacons, in the incoherent class at signal-to-noise densities of 26, 29, and 32 dB-Hz, resulted in location accuracies of 12.9, 5.6, and 7.7 km. These results show that the system requirements of 20 km accuracy is achievable, and that the detection thresholds for the coherent beacons are in agreement with the theoretical predictions given here.

VI. CONCLUSION

Frequency acquisition and measurement has been described from the point of view of the Cramer-Rao lower bound on frequency error. In the case of a single sinusoid or multiple well-separated sinusoids, a Fourier transform is a close approximation to an optimal acquisition algorithm. This is the first step, the coarse frequency estimator. The tradeoffs needed in order to select a proper window function have been described. The second step is a fine frequency estimator where it is shown that a simple and good method is to fit a Gaussian function to the FFT peak and its two neighbors. This method is able to achieve a frequency standard-deviation close to the Cramer-Rao limit and a frequency bias in the order of only a few percent of a bin. In the case of short-time stationary signals a proper compromise between FFT-length and number of averages has to be made in order to achieve the desired threshold SNR. Asymptotically a loss of 1.5 dB in threshold is encountered when coherent processing is traded for incoherent. We show here that for moderate number of averages and for an adaptive threshold detector, only between 0.5 and 1 dB is lost when this tradeoff is made.

The COSPAS-SARSAT system is used as an example throughout the paper. The ELT and EPIRB signals have been characterized in terms of degree of nonstationarity, and optimum DFT-based algorithms derived. A practical algorithm is then described. It is suitable for both coherent and incoherent beacons. It is based on a 1 s average of DFTs followed by a peak-picking detector that is matched to signal characteristics. The theoretical analysis has been compared with test results and good agreement is found. This algorithm is used in a local user terminal which has been in daily operational use in Tromsø, Northern Norway since the COSPAS-SARSAT system was launched in 1983. It started out as an experimental processing facility, but soon proved invaluable in locating emergencies. Today it acts as a front-line system in the Norwegian Search and Rescue Operation.

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REFERENCES

Air and sea rescue via satellite systems.  

Random error in ARGOS and SARSAT satellite  
positioning systems.  
*IEEE Transactions on Aerospace and Electronic Systems*,  
AES-21 (Nov. 1985).

Basic concepts in the processing of SARSAT signals.  
*IEEE Transactions on Aerospace and Electronic Systems*,  
AES-23 (Mar. 1987).

Identification techniques for SARSAT signals.  
*IEEE Transactions on Aerospace and Electronic Systems*,  
AES-23 (Mar. 1987).

Spectral analysis of ELF signals for SARSAT.  
*IEEE Transactions on Aerospace and Electronic Systems*,  
AES-23 (Sept. 1987).

A precise multipass method for satellite Doppler  
positioning.  

Modern spectral estimation: Theory and application.  

Sverre Holm (M’83) was born in Oslo, Norway in 1954. He received the M.S. and Ph.D. degrees in electrical engineering from the Norwegian Institute of Technology, Trondheim, Norway, in 1978 and 1982. From 1978 to 1984 he did research in speech coding and spectral estimation at the Electronics Research Laboratory at the Norwegian Institute of Technology, from 1984 to 1986 he taught electrical engineering at Yarmouk University in Irbid, Jordan, and from 1986 to 1990 he was employed by Informasjonskontroll AS, Asker, Norway doing development work in synthetic aperture radar processing and work for the European Space Agency on spectral estimation. Since 1990 he has been involved in medical ultrasound research and development at Vingmed Sound AS. Since 1989 he has also been an Adjunct Professor, until 1992 at the Norwegian Institute of Technology, and since then at the University of Oslo. His research interests are digital signal processing, spectral estimation, and ultrasound and radar imaging.

Dr. Holm is a member of the European Association for Signal Processing (EURASIP) and the Norwegian Society for Signal Processing (NORSIG) and served as Vice-President of NORSIG in 1987–1989.
Dear Dr. Holm,

With great interest I read your fine paper 'Optimum FFT-Based Frequency Acquisition...' in the IEEE Tr. AES Jan.93. In a perfect way it links theoretical treatise with practical applications, especially in frequency estimation via FFT, and thus will be of great value for engineers in the fields of communications and signal analysis.

Unfortunately I run into problems when applying your equ's (7) and (2) to get the curve for the CRLB in your Fig. 7 (solid line):

Assuming \( f_s = 51200 \) Hz, \( N = 1024 \) and Hamming window as in your example, with \( C/N_0 = 1000 \) (\( \approx 30 \) dBHz) I get from (7)

\[
\text{SNR} = \frac{C}{N} = \frac{1000}{51200/1024} = 14.7 \ (\approx 11.7 \text{ dB})
\]

Inserting this into (2) yields

\[
\sigma_f = \frac{51200\sqrt{12}}{2\pi\sqrt{1024(1024^2 - 1)}14.7} = 0.22 \text{ Hz}
\]

whereas your CRLB-curve in Fig. 7 starts with \( \sigma_f = 4.2 \) Hz at \( C/N_0 = 30 \) dBHz.
Moreover, following the equs (17) and (47) from RIFE and BOORSTYN 1974, I think the SNR in your equ.(2) should be multiplied by a factor of 2:

$$\sigma_t = \frac{\sqrt{12}}{2\pi T \sqrt{N(N^2 - 1)2SNR}} = 0.16 \text{ Hz}$$

Doing so, in RIFE and BOORSTYN the equs (17), (47) and the CRLB in their Fig.4 are in agreement, whereas the CRLB in your Fig.7 lies approximately a factor of $4.2/0.16 = 26$ higher than my calculation based on RIFE and BOORSTYN.

Since your simulation results in Fig.7 confirm the accompanying CRLB, I'm sure that my calculation is incorrect or that I have overlooked some extra condition.

I would be much obliged to you if you could find the time to give me a short answer in order to solve the discrepancies between my calculations and yours.

Yours sincerely

[Signature]
24 June 1993

Dear Dr. Höring,

Thanks for the comments on my paper on “Optimum FFT-Based Frequency Acquisition ...”. It is good to hear that someone really takes the time to go through the details like you have done.

First let me comment on the factor of \( \sqrt{2} \) larger value for the Cramer-Rao bound that I give compared to that of Rife and Boorstyn. It is there because I deal with real signals while they treat complex signals. When Kay in his 1988 book refers to the Rife and Boorstyn result, I think he states this clearer than they did originally.

Second, the difference between your value for \( \sigma_f \) and mine lies in the value for SNR. Eq. (7) gives the value for SNR in a single bin from the carrier to noise density and when inserted in (2) gives the result you have given in your example. This is the procedure I described in the text following (7).

I discovered after I went through my calculations again, that the value for SNR to be used in (2) should really be the input SNR, i.e. the SNR in the whole processed band, or \[ \text{SNR} = \frac{C}{N} = \frac{C}{N_0} \left( \frac{f_c}{2} \right) \], not the SNR inside the estimator that eq. (7) gives. Thus in the example, SNR=0.0039 or -14.1 dB, giving \( \sigma_f = 4.36 \text{ Hz} \).

Thank you very much for pointing out this short-coming of my paper.

Sincerely

Sverre Holm
Professor