Fourier-Bessel Analysis of Mean and Maximum Bessel Beam Quantisation in Equal-Area Annular Arrays

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Abstract : A Fourier-Bessel series is used to study the effects of quantising the J_0 Bessel beam in annular medical ultrasound transducers by two established methods. The results suggest that the key design properties are a function of the relative distribution of quantisation levels rather than in their absolute values. The analysis could form the basis for optimisation of the relative quantisation levels in future work.

1. Introduction

Bessel beams represent solutions to wave equation which have attractive properties for ultrasound imaging in that they have large depths of field (theoretically infinite). The most commonly studied beam is the J_0 Bessel beam, which in cylindrical coordinates has the form

$$p_0(r,z,t) = J_0(\alpha r)e^{j(\beta z - \omega t)} \qquad (1)$$

$J_0(\alpha r)$ is the Bessel function of the first kind of order zero, r is the radial distance from the cylindrical centerline and z is a distance perpendicular to the radial measure. Time is indicated by t and φ is the polar angle around the z axis. The α and β parameters are related via the wavenumber k

$$\beta = \sqrt{k^2 - \alpha^2} \qquad (2)$$

where $k = \omega/c$, ω is the frequency in rads/sec and c is the speed of sound in the medium. We observe in (1) that the Bessel beam is nondiffracting (has infinite depth of focus) since its radial profile $J_0(\alpha r)$ is independent of the distance z from the transducer surface. With the transducer placed at $z = 0$ and centred around $r = 0$, the (normalised) transducer surface pressure is $p_0(r,0,t) = J_0(\alpha r)e^{-j\omega t}$. For exact implementation, this expression requires a transducer with infinite radius and infinite radial resolution since $J_0(\alpha r)$ extends to infinity and is ever-changing in amplitude as a function of r. In practice the beams are approximated over a finite aperture by a discrete set of annular rings and they assume limited diffraction properties with finite depth of focus. In [1] and [2], the J_0 Bessel beam was approximated with its mean and maximum respectively value within each lobe of the Bessel function. Simulation studies and in-vitro experiments seem to indicate that this is a fairly good way to approximate the Bessel function, but the approximations were not backed by any theory. In [3] we introduced a methodology for analysing the quantisation in a theoretical manner; in this article we apply the method numerically to study the mean and maximum quantisation methods already established.

2. Summary of Fourier-Bessel Analysis

The Fourier-Bessel series expansion of order zero for a given function $f(r)$ is given in [4] as

$$f(r) = \sum_{i=1}^{\infty} A_i J_0(\alpha_i r) \qquad (3)$$

This is an orthogonal series in the range $0 < r < a$, where a is any desired value and α_i represent the infinite set of solutions to $J_0(\alpha_i a) = 0$. For annular transducers the A_i coefficients are given by

$$A_i = \frac{2\sum_p q_p\left[r_p^+ J_1(\alpha_i r_p^+) - r_p^- J_1(\alpha_i r_p^-)\right]}{\alpha_i a^2 J_1^2(\alpha_i a)} \qquad (4)$$

in which p represents the p^{th} annulus in the transducer, q_p represents the corresponding quantisation amplitude, and r_p^-, r_p^+ representing the inner and outer radii of the p^{th} annulus respectively. $J_1(.)$ represents the Bessel function of the first kind of order one. Since $k = \omega/c$ must remain fixed for all α_i to correctly implement the time envelope in (1), we have corresponding values $\beta_i = \sqrt{k^2 - \alpha_i^2}$ and depths of field (DOF) [5] for $\alpha_i \le k$ given by

$$DOF_i \simeq a\sqrt{(k/\alpha_i)^2 - 1} \quad , \quad \alpha_i \le k \qquad (5)$$

The beam components for $\alpha_i > k$ correspond to evanescent waves since β_i becomes a large imaginary value and $e^{-\beta_i z}$ decays to negligible levels within a few wavelengths of the transducer surface. The parameter a is chosen such that $a \ge R$ and coincides with a zero of the desired Bessel beam, i.e $J_0(\alpha a) = 0$. If this is the m^{th} zero of $J_0(\alpha r)$, then $\alpha_m = \alpha$ by definition and the desired beam is represented by $A_m = 1$ with $A_{i \ne m} = 0$. Due to the quantisation effects, this does not occur in practice however and we have the following conclusions from (2), (3) and (5). First, the quantised field is represented by an infinite sum of limited diffraction beams with effective aperture a. All beam components for $\alpha_i \le \alpha$ have a larger depth of focus than the desired beam, whilst the beam components for $\alpha < \alpha_i \le k$ have smaller depth of focus. The component corresponding to $\alpha_m = \alpha$ represents the beam component corresponding to the desired beam. Values for $\alpha_i > k$ represent evanescent waves which may be neglected.

3. Application to Equal Area Transducer

We now apply the method to Bessel beam quantisation in an equal area transducer as detailed in [2]. The radius of the first annulus is placed to coincide with the first zero of the desired Bessel function and the remaining radii are selected such that each remaining annulus has equal area. The quantisation levels in [2] were chosen as the mean values of the desired Bessel function over each annulus; we study this case here and also a further design in which the maximum amplitudes are taken as the quantisation levels. Consider a desired Bessel beam, with frequency 3.5MHz and parameter $\alpha = 641.2869\ m^{-1}$. Assuming a medium with velocity of sound 1540 m/s, the corresponding wavenumber is $k = 14279.97\ m^{-1}$ and this indicates the α_i limit before evanescence. The first zero of $J_0(.)$ may be found numerically as 2.404826, which gives the radius of the first annulus as $r = 3.75$mm ($\alpha r = 2.404826$). For an equal area four-annulus design this gives $r_1^- = 0$mm, $r_1^+ = 3.75$mm, $r_2^- = 3.75$mm, $r_2^+ = 5.30$mm $= r_3^-$, $r_3^+ = 6.50$mm $= r_4^-$ and $r_4^+ = 7.50$mm. Figure 1 shows the resultant radial profile of the transducer, with ideal, mean-design and maximum-design beam profiles.

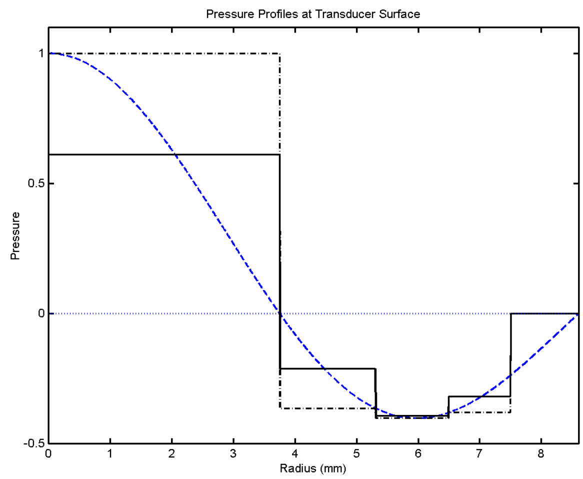

Figure 1 Pressure profiles : Ideal Bessel (curved), Mean quantised (solid), Maximum quantised (dashed)

Note that the outer radius $R = r_4^+ = 7.5$mm does not coincide with a zero of the desired Bessel function, and we select the a parameter in (4) as $a = 8.6078$mm such that it then coincides with the next Bessel zero ($m = 2$) at $\alpha_2 a = 5.520078$. Evaluation of further Bessel zeros shows that the first 39 α_i coefficients fall within the evanescence limit and the corresponding A_i coefficients are shown in Figure 2.

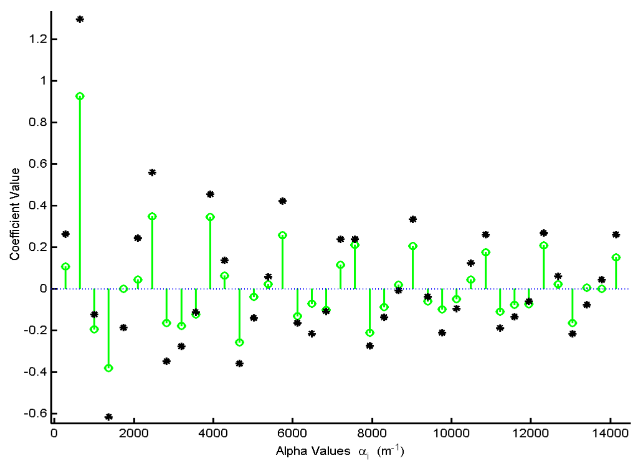

Figure 2 Coefficients A_i for $i = 1...39$: Mean quantised (lines ending 'o'), Maximum quantised (stars '*')

The values A_2 in second position along the α_i axis correspond to the location of the desired beam ($m = 2$, $\alpha_m = \alpha = 641.2869\ m^{-1}$). Thus α_1 corresponds to a beam component with greater depth of field than the required beam whereas α_3 to α_{39} correspond to beam components with shorter depth of field. We also see that the mean design produces less coefficient deviation from the ideal beam $A_2 = 1$, $A_{i \neq 2} = 0$ than the maximum design. In particular, the mean design has slightly less than the desired beam component at $\alpha_2 = \alpha$, whereas the maximum has considerably more. The quantisation levels of each design may therefore be rescaled by a factor $q_p \rightarrow q_p / A_m$ to force both beams to transmit the desired amplitude $A_2 = 1$ at $\alpha_2 = \alpha$ if desired. However, the mean design still performs better after the rescaling, with a sum of squared coefficient errors relative to the ideal beam 24.45% larger for the maximum design than for the mean design.

4. Conclusions

We have applied the Fourier-Bessel method introduced in [4] to an equal area 4-annulus transducer with given design parameters. The mean design was superior in terms of coefficient error distribution in Fourier-Bessel space. The numerical procedures involved in the analysis are easy to implement and allow us to determine exactly all the limited diffraction components representing the quantised Bessel beam profile. This enables us perform a rescaling of the quantisation levels to specify a particular amount of power in any chosen beam component if desired. Future work might lead to superior quantisation designs by optimising the coefficient distribution in a suitable manner; for example a least squares fit between actual and desired coefficients, a constrained minimisation of $\max|A_{i \neq m}|$ subject to $A_m = 1$, or a total nullification of particular beam coefficients.

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6. References

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