

Asymptotic Convergence of Degree-Raising

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It is well known that the degree-raised Bernstein-Bézier coefficients of degree n of a polynomial g converge to g at the rate $1/n$. In this paper we consider the polynomial $A_n(g)$ of degree $\leq n$ interpolating the coefficients. We show how A_n can be viewed as an inverse to the Bernstein polynomial operator and that the derivatives $A_n(g)^{(r)}$ converge uniformly to $g^{(r)}$ at the rate $1/n$ for all r . We also give an asymptotic expansion of Voronovskaya type for $A_n(g)$ and discuss some shape preserving properties of this polynomial.

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1. Introduction

Suppose g is a polynomial, of degree say d , $g \in \pi_d$, and let $n \geq d$. In terms of the Bernstein basis for polynomials of degree n given by

$$p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}, \quad \nu = 0, \dots, n, \quad (1.1)$$

we can represent g in *degree-raised Bernstein form* as

$$g = \sum_{\nu=0}^n a_{n,\nu} p_{n,\nu}, \quad (1.2)$$

where the $a_{n,\nu}$ are called the Bernstein/Bézier coefficients (BB-coefficients) of g of degree n .

This form of *degree-raising* has been considered by many authors, see [3,5,9–11,15] and references therein. Degree-raising is used in computer aided geometric design to model a curve by its *control polygon* $P_n(g)$, i.e. the piecewise

linear function connecting the *control points* $(\nu/n, a_{n,\nu})$ for $\nu = 0, \dots, n$. The curve inherits the shape of $P_n(g)$ ([6]) and it has been shown in [11] that the degree-raised BB-coefficients $a_{n,\nu}$ of g converge to g at the rate $1/n$,

$$a_{n,\nu} - g\left(\frac{\nu}{n}\right) = O\left(\frac{1}{n}\right), \quad n \rightarrow \infty. \quad (1.3)$$

This implies that the control polygon $P_n(g)$ converges uniformly to g as n tends to infinity.

The control polygon is only one of many curves which interpolates or approximates the control points and convergence can perhaps be better analyzed by considering a different approximation. In this paper we consider the polynomial $A_n(g)$ of degree $\leq n$ passing through the control points. In [15] this approach was used to show $1/n$ convergence of $A_n(g)$ to g . Since $A_n(g)$ is differentiable we can study the convergence of the derivatives of $A_n(g)$ to the corresponding derivatives of g . We do this in Section 2, where we show that all the derivatives of $A_n(g)$ converge to the derivatives of g at the same rate $1/n$. Using A_n we are also able to give a refined convergence analysis of degree-raising. In fact we derive in Section 3 an asymptotic expansion of Voronovskaya type ([1,16]) for A_n . The main tool is a simple three term recurrence relation for certain coefficients of A_n . Finally in Section 4 we study shape preserving properties of A_n .

After this paper was completed we discovered that the recurrence relation for A_n has recently been shown in [7] and [14].

2. Convergence of Derivatives

Degree-raising and approximation by Bernstein polynomials are closely related. In fact, g is the Bernstein polynomial of the control polygon $f = P_n(g)$ of g . In symbols $g = B_n(P_n(g))$, where for any continuous function f on $[0, 1]$

$$B_n(f) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) p_{n,\nu}. \quad (2.1)$$

Thus the operator P_n is a one-sided inverse of the Bernstein operator B_n . Suppose that $g \in \pi_n$ is given by (1.2) and let $A_n(g)$ be the polynomial of degree $\leq n$ interpolating the degree n control points $(\nu/n, a_{n,\nu})_{\nu=0}^n$ of g . Since $B_n(f)$ only depends on the values of f at the points $(\nu/n)_{\nu=0}^n$ it follows that also A_n is a

one-sided inverse of B_n . Moreover, A_n is the unique inverse of B_n as an operator on π_n . The Lagrange form of $A_n(g)$ is

$$A_n(g) = \sum_{\nu=0}^n a_{n,\nu} \ell_{n,\nu}, \quad (2.2)$$

where $\ell_{n,\nu} \in \pi_n$ is the Lagrange polynomial defined by $\ell_{n,\nu}(\mu/n) = \delta_{\mu\nu}$ for $\mu = 0, \dots, n$.

It is well known that if $f \in \pi_d$ for some $d \leq n$, then also $B_n(f) \in \pi_d$. This follows from the formula

$$B_n(f, x) = \sum_{k=0}^n \binom{n}{k} \Delta^k f(0) x^k,$$

which can be found for example in Chapter 10 of [4]. Here Δ^k denotes powers of the usual forward difference with spacing $1/n$ given by $\Delta f(0) = f(1/n) - f(0)$. It follows that if $g \in \pi_d$ then $A_n(g) \in \pi_d$.

Let $g \in \pi_d$ be given. To give a representation for $A_n(g)$ which shows more clearly that it is a polynomial of degree $\leq d$ it is convenient to introduce factorial notation. For any x and any nonnegative integer n we let $x^{(0)_n} = x^{(0)} = 1$ and for $m \geq 1$

$$x^{(m)_n} = x^{(m)} := x(x - \frac{1}{n}) \cdots (x - \frac{m-1}{n}) \quad (2.3)$$

be the m th degree factorial with spacing $1/n$. The factorial is related to a binomial coefficient by the formula

$$x^{(m)_n} = n^{-m} m! \binom{nx}{m}, \quad (2.4)$$

where for all $y \in \mathbb{R}$, $\binom{y}{m} = y(y-1) \cdots (y-m+1)/m!$ for $m \geq 0$ and we define $\binom{y}{m} = 0$ for $m < 0$.

Using this notation, we next define for integers $n \geq d \geq 0$,

$$p_{d,\nu,n}(x) = \binom{d}{\nu} x^{(\nu)} (1-x)^{(d-\nu)} / 1^{(d)}, \quad \nu = 0, \dots, d, \quad x \in \mathbb{R}, \quad (2.5)$$

where the factorials $x^{(\nu)}$, $(1-x)^{(d-\nu)}$, and $1^{(d)}$ have a spacing of $1/n$ as in (2.3). Observe that $p_{d,\nu,n}(x)$ is a polynomial of degree d in x , with the d roots $0, 1/n, \dots, (\nu-1)/n, 1 - (d-\nu-1)/n, \dots, 1 - 1/n, 1$, and it is non-negative in the interval $[(\nu-1)/n, 1 - (d-\nu-1)/n]$. This implies that any non-negative linear combination of the $p_{d,\nu,n}$ is non-negative in the interval $[(d-1)/n, 1 - (d-1)/n]$,

for $n \geq 2d - 2$. We call the function $p_{d,\nu,n}$ a *discrete Bernstein polynomial*. The first specific study of such functions was carried out in [9,10] and was later considered in [13].

From (2.4) we immediately obtain the alternative representation

$$p_{d,\nu,n}(x) = \binom{nx}{\nu} \binom{n(1-x)}{d-\nu} / \binom{n}{d}. \quad (2.6)$$

We have the following representation for $A_n(g)$ in terms of the polynomials $p_{d,\nu,n}$.

Proposition 2.1. Let $n \geq d \geq 0$ and $g \in \pi_d$. Then

$$A_n(g) = \sum_{\nu=0}^d a_{d,\nu} p_{d,\nu,n}, \quad (2.7)$$

where the $a_{d,\nu}$ are the BB-coefficients of g of degree d .

Proof. The degree n BB-coefficients $(a_{n,\nu})_{\nu=0}^n$ can be expressed in terms of the degree d BB-coefficients as follows ([5])

$$a_{n,k} = \sum_{\nu=0}^d a_{d,\nu} \frac{\binom{d}{\nu} \binom{n-d}{k-\nu}}{\binom{n}{k}}, \quad k = 0, \dots, n. \quad (2.8)$$

But using (2.6) and rearranging terms we find

$$\sum_{\nu=0}^d a_{d,\nu} p_{d,\nu,n} \left(\frac{k}{n}\right) = \sum_{\nu=0}^d a_{d,\nu} \frac{\binom{k}{\nu} \binom{n-k}{d-\nu}}{\binom{n}{d}} = \sum_{\nu=0}^d a_{d,\nu} \frac{\binom{d}{\nu} \binom{n-d}{k-\nu}}{\binom{n}{k}} = a_{n,k}. \quad (2.9)$$

Substituting this into (2.2) we obtain

$$A_n(g) = \sum_{k=0}^n \sum_{\nu=0}^d a_{d,\nu} p_{d,\nu,n}(k/n) \ell_{n,k}.$$

Switching the order of summation and observing that

$$p_{d,\nu,n} = \sum_{k=0}^n p_{d,\nu,n}(k/n) \ell_{n,k}$$

we obtain (2.7). □

Consider now convergence of $A_n(g)$ to g . We first show that the derivatives $D^k p_{d,\nu}(x)$ and $D^k p_{d,\nu,n}(x)$ are $O(1/n)$ perturbations of each other.

Lemma 2.2. Let d and n be integers with n positive and $0 \leq d \leq n$. There is a constant K depending only on d such that

$$|D^k p_{d,\nu,n}(x) - D^k p_{d,\nu}(x)| \leq \frac{K}{n}, \quad x \in [0, 1], \quad 0 \leq \nu, k \leq d. \quad (2.10)$$

Proof. By (1.1) and (2.5) we can write

$$\begin{aligned} & D^k p_{d,\nu,n}(x) - D^k p_{d,\nu}(x) \\ &= \binom{d}{\nu} \left[D^k x^{(\nu)} (1-x)^{(d-\nu)} - 1^{(d)} D^k x^\nu (1-x)^{d-\nu} \right] / 1^{(d)} \\ &= \binom{d}{\nu} \left[D^k (x^{(\nu)} - x^\nu) (1-x)^{(d-\nu)} \right. \\ &\quad \left. + D^k x^\nu ((1-x)^{(d-\nu)} - (1-x)^{d-\nu}) + D^k x^\nu (1-x)^{d-\nu} (1 - 1^{(d)}) \right] / 1^{(d)}. \end{aligned} \quad (2.11)$$

For a positive integer r , any x_1, \dots, x_r with $0 \leq x_i \leq d/n$, and any $x \in [0, 1]$ we have

$$\begin{aligned} |(x - x_1) \cdots (x - x_r) - x^r| &\leq \frac{rd}{n}, \\ |(x - x_1) \cdots (x - x_r)| &\leq 1, \\ 1^{(d)n} &\geq 1^{(d)d} = (d-1)! / d^{d-1}. \end{aligned} \quad (2.12)$$

The last inequality in (2.12) is obvious, while the second follows since each factor is bounded by one. Finally, the first inequality clearly holds for $r = 1$ and if it holds for $r - 1$ then for $x \in [0, 1]$,

$$\begin{aligned} |(x - x_1) \cdots (x - x_r) - x^r| &= |(x - x_r)((x - x_1) \cdots (x - x_{r-1}) - x^{r-1}) - x_r x^r| \\ &\leq |(x - x_1) \cdots (x - x_{r-1}) - x^{r-1}| + x_r \\ &\leq \frac{(r-1)d}{n} + \frac{d}{n} = \frac{rd}{n}. \end{aligned}$$

To complete the proof we differentiate the products in (2.11) and invoke the triangle inequality. By using the inequalities (2.12) on each term the result follows. \square

We can now show that derivatives of $A_n(g)$ converge to corresponding derivatives of g .

Theorem 2.3. For each nonnegative integer d and $g \in \pi_d$ there is a constant C such that for all $n \geq d$

$$|D^k A_n(g, x) - D^k g(x)| \leq \frac{C}{n}, \quad x \in [0, 1], \quad 0 \leq k \leq d. \quad (2.13)$$

Proof. By (1.2) with $n = d$, (2.7), and Lemma 2.2

$$|D^k A_n(g, x) - D^k g(x)| \leq \sum_{\nu=0}^d |a_{d,\nu}| |D^k p_{d,\nu,n}(x) - D^k p_{d,\nu}(x)| \leq \frac{K}{n} \sum_{\nu=0}^d |a_{d,\nu}|,$$

where K is the constant in Lemma 2.2. The result now follows. \square

3. Asymptotic Convergence of Degree-Raising

It was shown by Voronovskaya [16] that if $f \in C[0, 1]$, then

$$\lim_{n \rightarrow \infty} n(B_n(f, x) - f(x)) = \frac{1}{2}x(1-x)f''(x),$$

where B_n is the Bernstein operator defined in (2.1). Bernstein [1] (see also [8]) later generalized this to an asymptotic formula involving higher derivatives of f , provided f is smooth enough. In particular when $f \in \pi_d$, $d \leq n$, the formula reduces to

$$\lim_{n \rightarrow \infty} n^q \left[B_n(f, x) - f(x) - \sum_{\mu=2}^{2q-1} \frac{1}{\mu!} \frac{1}{n^\mu} T_{n,\mu}(x) f^{(\mu)}(x) \right] = \left(\frac{1}{2}x(1-x)\right)^q \frac{1}{q!} f^{(2q)}(x), \quad (3.1)$$

for $q \leq (d+1)/2$, where

$$T_{n,\mu}(x) = \sum_{\nu=0}^n (\nu - nx)^\mu p_{n,\nu}(x).$$

Equation (3.1) is derived in [8] from the well-known recurrence formula,

$$T_{n,\mu+1}(x) - x(1-x)(T'_{n,\mu}(x) - \mu n T_{x,\mu-1}(x)) = 0,$$

and the identities

$$T_{n,0} = 1, \quad T_{n,1} = 0.$$

In this section we wish to obtain an asymptotic formula similar to (3.1) for the inverse Bernstein operator A_n and as a consequence degree-raising.

Proposition 3.1. Let $g \in \pi_d$ and $n \geq d$. Then

$$A_n(g)(x) = \sum_{\mu=0}^d \frac{(n-\mu)!}{n!} S_{n,\mu}(x) \frac{g^{(\mu)}(x)}{\mu!}, \quad x \in \mathbb{R}, \quad (3.2)$$

where

$$S_{n,\mu}(x) = \sum_{k=0}^{\mu} s_{n,\mu,k}(x) \quad (3.3)$$

and

$$s_{n,\mu,k}(x) = (-1)^{\mu-k} \binom{\mu}{k} x^{\mu-k} \frac{(n-k)!}{(n-\mu)!} nx(nx-1) \dots (nx-k+1). \quad (3.4)$$

Proof. It is well known that the $p_{n,\nu}$ are the B-splines of degree n over the knot vector $t_0 = \dots = t_n = 0$, $t_{n+1} = \dots = t_{2n+1} = 1$. Specializing the de Boor-Fix formula (see [2], p. 159) to this case one finds that the n th degree BB-coefficients $a_{n,\nu}$ of g satisfy

$$a_{n,\nu} = \sum_{\mu=0}^d (-1)^{\mu} \psi_{\nu,n}^{(n-\mu)}(\nu/n) g^{(\mu)}(\nu/n), \quad (3.5)$$

where

$$\psi_{\nu,n}(x) = x^{n-\nu}(x-1)^{\nu}/n!.$$

Expanding $\psi_{\nu,n}(x)$ in powers of x we obtain

$$\psi_{\nu,n}(x) = \frac{1}{n!} \sum_{k=0}^{\nu} (-1)^k \binom{\nu}{k} x^{n-\nu+k},$$

and differentiating $n - \mu$ times we find

$$\psi_{\nu,n}^{(n-\mu)}(x) = \frac{1}{n!} \sum_{k=\max(0,\nu-\mu)}^{\nu} (-1)^k \binom{\nu}{k} \frac{(n-\nu+k)!}{(\mu-\nu+k)!} x^{\mu-\nu+k}.$$

Now changing the summation index k to $\nu - k$ and removing a factor of $(n - \mu)!/\mu!$ one obtains

$$\begin{aligned} (-1)^{\mu} \psi_{\nu,n}^{(n-\mu)}(\nu/n) &= \frac{1}{n!} \sum_{k=0}^{\min(\nu,\mu)} (-1)^{\mu-k} \binom{\nu}{\nu-k} \frac{(n-k)!}{(\mu-k)!} (\nu/n)^{\mu-k} \\ &= \frac{(n-\mu)!}{\mu! n!} \sum_{k=0}^{\min(\nu,\mu)} (-1)^{\mu-k} \binom{\mu}{k} \frac{\nu!}{(\nu-k)!} \frac{(n-k)!}{(n-\mu)!} (\nu/n)^{\mu-k} \end{aligned}$$

$$= \frac{(n-\mu)!}{\mu! n!} S_{n,\mu}(\nu/n).$$

Combining this with (3.5), we find

$$B_n \left(\sum_{\mu=0}^d \frac{(n-\mu)!}{\mu! n!} S_{n,\mu} g^{(\mu)} \right) = \sum_{\nu=0}^d a_{n,\nu} p_{n,\nu} = g,$$

and (3.4) follows from applying A_n . \square

It was noted in [12] that $A_n(g)$ can be expressed as a linear combination of derivatives of g and this observation was used as the start point for studying certain quasi-interpolant operators [12,17].

Now we take a closer look at the functions $S_{n,\mu}$. From (3.3) the first three polynomials in the sequence are $S_{n,0}(x) = 1$, $S_{n,1}(x) = 0$, and $S_{n,2}(x) = -nx(1-x)$. In order to compute further members of the sequence and to derive further facts about $S_{n,\mu}$ a three-term recurrence formula will now be derived.

Lemma 3.2. For all x , all n and $\mu = 1, 2, 3, \dots, n-1$,

$$S_{n,\mu+1}(x) + \mu(1-2x)S_{n,\mu}(x) + \mu(n-\mu+1)x(1-x)S_{n,\mu-1}(x) = 0. \quad (3.6)$$

Proof. From (3.4) we observe that

$$x s_{n,\mu,k} = -\frac{(\mu-k+1)}{(n-\mu)(\mu+1)} s_{n,\mu+1,k} \quad (3.7)$$

and

$$(nx-k)s_{n,\mu,k} = \frac{(n-k)(k+1)}{(n-\mu)(\mu+1)} s_{n,\mu+1,k+1}. \quad (3.8)$$

Define $s_{n,\mu,k} = 0$ for $k > \mu$. Then, firstly applying (3.7) with μ replaced by $\mu-1$ and secondly applying both (3.7) and (3.8) we obtain

$$\begin{aligned} & S_{n,\mu+1} + \mu(1-2x)S_{n,\mu} + \mu(n-\mu+1)x(1-x)S_{n,\mu-1} \\ &= \sum_{k=0}^{\mu+1} (s_{n,\mu+1,k} + \mu(1-2x)s_{n,\mu,k} + \mu(n-\mu+1)x(1-x)s_{n,\mu-1,k}) \\ &= \sum_{k=0}^{\mu+1} (s_{n,\mu+1,k} + \mu(1-2x)s_{n,\mu,k} - (\mu-k)(1-x)s_{n,\mu,k}) \\ &= \sum_{k=0}^{\mu+1} (s_{n,\mu+1,k} + (n-k-\mu)x s_{n,\mu,k} - (nx-k)s_{n,\mu,k}) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(n-\mu)(\mu+1)} \sum_{k=0}^{\mu+1} (k(n-k+1)s_{n,\mu+1,k} - (k+1)(n-k)s_{n,\mu+1,k+1}) \\
 &= 0.
 \end{aligned}$$

□

Using (3.6) it is straightforward to compute further polynomials in the sequence. Let $X = x(1-x)$ and $Y = 1-2x$. Then noting that $Y^2 = 1-4X$ we find

$$\begin{aligned}
 S_{n,0} &= 1, & S_{n,1} &= 0, & S_{n,2} &= -nX, & S_{n,3} &= 2nYX, \\
 S_{n,4} &= 3nX[(n+6)X-2], & S_{n,5} &= -4nYX[(5n+12)X-6], \\
 S_{n,6} &= -5nX[(3n^2+86n+120)X^2 - (26n+120)X + 24].
 \end{aligned}$$

Equation (3.4) shows that $S_{n,\mu}$ is a polynomial in n as well as x but (3.6) gives us the following more precise statement.

Lemma 3.3. The polynomials $S_{n,2\mu}$ and $S_{n,2\mu+1}$ can be expressed as

$$S_{n,2\mu}(x) = \sum_{j=0}^{\mu} b_{j,2\mu}(X)n^j, \quad S_{n,2\mu+1}(x) = Y \sum_{j=0}^{\mu} b_{j,2\mu+1}(X)n^j, \quad (3.9)$$

where $b_{j,2\mu}$ and $b_{j,2\mu+1}$ are polynomials of degree $\leq \mu$ and

$$b_{\mu,2\mu}(X) = (-1)^\mu 1 \cdot 3 \cdot 5 \cdots (2\mu-1)X^\mu = (-1)^\mu \frac{(2\mu)!}{2^\mu \mu!} X^\mu. \quad (3.10)$$

Proof. By induction on μ using (3.6) and the identities $S_{n,0} = 1$ and $S_{n,1} = 0$, the expressions (3.9) are easily established once one sees the identity $Y^2 = 1-4X$.

To obtain an expression for the highest order coefficient $b_{\mu,2\mu}(X)$ in $S_{n,2\mu}$, we observe from (3.6) that

$$S_{n,2\mu}(x) = -(2\mu-1)Y S_{n,2\mu-1}(x) - (2\mu-1)(n-2\mu+2)X S_{n,2\mu-2}(x),$$

and it follows that

$$b_{\mu,2\mu}(X) = -(2\mu-1)X b_{\mu-1,2\mu-2}(X).$$

Then, because $b_{1,2}(X) = -X$, equation (3.10) follows. □

We now apply Lemma 3.3 to obtain an expansion similar to (3.1).

Theorem 3.4. Let $g \in \pi_d$. Then for $q = 1, 2, 3, \dots, [d/2]$,

$$\begin{aligned} \lim_{n \rightarrow \infty} n^q \left[A_n(g, x) - g(x) - \sum_{\mu=2}^{2q-1} \frac{(n-\mu)!}{\mu! n!} S_{n,\mu}(x) g^{(\mu)}(x) \right] \\ = (-1)^q \left(\frac{1}{2}X\right)^q \frac{1}{q!} g^{(2q)}(x), \end{aligned} \quad (3.11)$$

and the limit is uniform for $x \in [0, 1]$.

Proof. Noting that

$$(n-\mu)!/n! = 1/(n(n-1)\dots(n-\mu+1)) = O(n^{-\mu})$$

and applying the expression for $b_{\mu,2\mu}$ in (3.10), we deduce that

$$\begin{aligned} n^q \sum_{\mu=2q}^d \frac{(n-\mu)!}{\mu! n!} S_{n,\mu} g^{(\mu)} \\ = \frac{1}{(2q)!} b_{q,2q}(X) g^{(2q)} + O\left(\frac{1}{n}\right) = (-1)^q \left(\frac{1}{2}X\right)^q \frac{1}{q!} g^{(2q)} + O\left(\frac{1}{n}\right). \end{aligned}$$

□

As an example, letting $q = 1$ in (3.11), we obtain

$$\lim_{n \rightarrow \infty} n[A_n(g, x) - g(x)] = -\frac{1}{2}x(1-x)g''(x).$$

In order to apply (3.11) to the convergence of the degree-raised BB-coefficients, we only have to observe from (2.3) that $A_n(g, \nu/n) = a_{\nu,n}$. Therefore since the right hand side of (3.11) is bounded for $x \in [0, 1]$, we finally arrive at a generalization of equation (1.3).

Corollary 3.5. Under the hypothesis of Theorem 3.4

$$a_{n,\nu} - g\left(\frac{\nu}{n}\right) - \sum_{\mu=2}^{2q-1} \frac{(n-\mu)!}{\mu! n!} S_{n,\mu}\left(\frac{\nu}{n}\right) g^{(\mu)}\left(\frac{\nu}{n}\right) = O\left(\frac{1}{n^q}\right), \quad n \rightarrow \infty,$$

and the bound is uniform.

4. Shape Preservation

It was shown in [6] that the sequence $\{P_n(g)\}_n$ is *variation diminishing* in the sense that $P_{n+1}(g)$ has no more zeros in the interval $[0, 1]$ than $P_n(g)$.

The sequence $\{P_n(g)\}_n$ enjoys several other shape preserving properties such as positivity, monotonicity, and convexity. In this section we study to what extent the sequence $\{A_n(g)\}_n$ is also shape preserving.

Considering the case $n = d + 1$ in (2.8), we obtain the well known degree-raising identity

$$a_{d+1,\nu} = \frac{\nu}{d+1}a_{d,\nu-1} + \frac{d+1-\nu}{d+1}a_{d,\nu}. \quad (4.1)$$

It immediately follows that

$$A_{d+1}(g, x) = xA_d\left(g, \frac{(d+1)x-1}{d}\right) + (1-x)A_d\left(g, \frac{(d+1)x}{d}\right), \quad (4.2)$$

due to the evident fact that the two polynomials on either side of this equation have degree $\leq d + 1$ and agree at the points $x = 0, 1/(d + 1), \dots, 1$. It follows that there is partial positivity preservation in the sense that if $A_d(g, x) \geq 0$ for all $x \in [0, 1]$, then $A_{d+1}(g, x) \geq 0$ for all $x \in [1/(d + 1), d/(d + 1)]$. By strengthening equation (4.2), we can obtain a more general positivity property.

Proposition 4.1. Let $g \in \pi_d$. Then for $n \geq d$,

$$A_n(g, x) = \sum_{\nu=0}^{n-d} A_d\left(g, \frac{nx-\nu}{d}\right)p_{n-d,\nu,n}(x) \quad (4.3)$$

Proof. From equation (2.9), we have

$$a_{n,k} = \sum_{\nu=0}^d a_{d,\nu} \frac{\binom{k}{\nu} \binom{n-k}{d-\nu}}{\binom{n}{d}}.$$

Then, since $a_{m,\nu} = A_m(g, \nu/m)$ for all ν and m , and replacing ν by $k - \nu$, we have

$$A_n\left(g, \frac{k}{n}\right) = \sum_{\nu=0}^{n-d} A_d\left(g, \frac{k-\nu}{d}\right) \frac{\binom{k}{\nu} \binom{n-k}{n-d-\nu}}{\binom{n}{d}},$$

and so, from (2.4),

$$A_n\left(g, \frac{k}{n}\right) = \sum_{\nu=0}^{n-d} A_d\left(g, \frac{k-\nu}{d}\right) \binom{n-d}{\nu} \left(\frac{k}{n}\right)^{\nu} \left(1 - \frac{k}{n}\right)^{(n-d-\nu)} / 1^{(n-d)}.$$

It follows that equation (4.3) holds for $x = 0, 1/n, \dots, 1$, and therefore for all x , since both sides of the equation are polynomials of degree $\leq n$. \square

Corollary 4.2. For $d \leq n \leq 2d$, if $A_d(g, x) \geq 0$ for all $x \in [0, 1]$, then $A_n(g, x) \geq 0$ for all $x \in [1 - d/n, d/n]$.

Proof. We know that the polynomials $p_{n-d, \nu, n}(x)$ are non-negative for x in the interval $[(n - d - 1)/n, 1 - (n - d - 1)/n]$ which is contained in the interval $[1 - d/n, d/n]$ in which $A_d(g, (nx - \nu)/d)$ is non-negative for all $\nu = 0, \dots, n - d$. The result now follows from (4.3). \square

However non-negativity of $A_d(g)$ in $[0, 1]$ does not guarantee non-negativity of $A_{d+1}(g)$ in the whole of $[0, 1]$. For example, letting

$$g(x) = \frac{x + 16x^2 + 8x^3}{36},$$

one finds that

$$A_3(g, x) = x(x - 1/6)^2,$$

which is nonnegative on $[0, 1]$. From equation (4.2) however, we have that

$$A_4(g, x) = \frac{-5x + 16x^2 + 64x^3}{108}$$

and so $A_4(g, 1/8) = -1/432$.

Corollary 4.2 is only valid for $n \leq 2d$. However from Proposition 2.1 we also have

Proposition 4.3. For $d \geq 2$ and $n \geq 2d - 2$, if $A_d(g) \geq 0$ in $[0, 1]$, then $A_n(g, x) \geq 0$ for all $x \in [(d - 1)/n, 1 - (d - 1)/n]$.

Finally, there appears to be no partial monotonicity or convexity preservation in the spirit of Corollary 4.2. One can show that if

$$g(x) = \frac{481x - 900x^2 + 840x^3 - 360x^4 + 144x^5}{18750},$$

then

$$A_5(g, x) = \int_0^x (y - 2/5)^2 (y - 3/5)^2 dy,$$

so that $DA_5(g, x) \geq 0$ for all x in $[0, 1]$ and $A_5(g)$ is thus monotonically non-decreasing there. However, computing $A_6(g)$ and differentiating, one finds that

$$DA_6(g, 1/2) = -1/93750 < 0,$$

and so $A_6(g)$ is decreasing in the centre of the interval $[0, 1]$.

Similarly, if one takes

$$g(x) = \frac{637x + 935x^2 - 1200x^3 + 900x^4 + 240x^6}{466560},$$

then

$$A_6(g, x) = \int_0^x \int_0^y (z - 1/3)^2 (z - 1/2)^2 dz, dy$$

and so $D^2 A_6(g, x) \geq 0$ for all x in $[0, 1]$ and thus $A_6(g)$ is convex on $[0, 1]$. However the same property does not hold for A_7 since by computing it and differentiating it twice one deduces that

$$D^2 A_7(g, 3/7) = -7/699840 < 0.$$

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