On a Class of Weak Tchebycheff Systems

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Abstract. In this paper we study the approximation power, the existence of a normalized B-basis and the structure of a degree-raising process for spaces of the form

$$\text{span} < 1, x, \dots, x^{n-2}, u(x), v(x) >,$$

requiring suitable assumptions on the functions u and v. The results about degree raising are detailed for special spaces of this form which have been recently introduced in the area of CAGD.

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§1. Introduction and Preliminaries

Recently, in the area of CAGD the following spaces have been studied (see [2], [11], [17] and references quoted therein)

$$\text{span} < 1, x, \cos(x), \sin(x) >, \quad x \in [0, \alpha], \ \alpha < \pi, \tag{1}$$

$$\text{span} < 1, x, x^2, \cos(x), \sin(x) >, \quad x \in [0, \alpha], \ \alpha < \pi, \tag{2}$$

and

$$\text{span} < 1, x, (1-x)^{m_0}, x^{m_1} >, \quad x \in [0, 1], \ 3 \le m_0, m_1 \in \mathbb{N}. \tag{3}$$

These spaces provide interesting performances, mainly related to the possibility of controlling the shape of their elements. The shape parameters are α for spaces (1) and (2) and m_0, m_1 for the space (3).

This fact motivates the present paper where we study the approximation power, the existence of a normalized B-basis and the structure of a degree-raising process for the space

$$\mathcal{P}^n_{u,v}([a, b]) := \text{span} < 1, x, \dots, x^{n-2}, u(x), v(x) >, \ x \in [a, b], \ 2 \le n \in \mathbb{N}. \tag{4}$$

We assume that $u, v \in C^{n+1}([a, b])$ and

$$\dim \left(\mathcal{P}^n_{u,v}([a, b]) \right) = n + 1, \tag{5}$$

so that $u^{(n-1)}$ and $v^{(n-1)}$ are linearly independent.

All the results we will obtain are based on the properties of the zeros of the derivative of order $n-1$ of the elements of $\mathcal{P}^n_{u,v}([a,b])$. Note that the elements spanning spaces (1) and (2) form Extended Tchebycheff systems on $[0,1]$ (see Definition 22) while those spanning the space (3) form only a Tchebycheff system on the closed interval $[0,1]$ (unless $m_0 = m_1 = 3$) (see Definition 17). This is due to the existence of too many zeros at the endpoints of the interval.

The following two conditions will be central in this paper. Let ψ be any element of $\mathcal{P}^n_{u,v}([a,b])$.

$$\begin{gathered}\text{If } \psi^{(n-1)}(x_1) = \psi^{(n-1)}(x_2) = 0,\ x_1, x_2 \in [a,b],\ x_1 \neq x_2, \\ \text{then } \psi^{(n-1)}(x) = 0,\ x \in [a,b].\end{gathered} \tag{6}$$

$$\begin{gathered}\text{If } \psi^{(n-1)}(x_1) = \psi^{(n)}(x_1) = 0,\ x_1 \in (a,b), \\ \text{then } \psi^{(n-1)}(x) = 0,\ x \in [a,b].\end{gathered} \tag{7}$$

We notice that, considering (5), (6) is the same as saying that $\{u^{(n-1)}, v^{(n-1)}\}$ is a Tchebycheff system in $[a,b]$ while (6) and (7) state that $\{u^{(n-1)}, v^{(n-1)}\}$ is an Extended Tchebycheff system in (a,b).

The class of spaces $\mathcal{P}^n_{u,v}([a,b])$ includes not only the spaces (3), and the spaces (1), (2) and their generalization

$$\text{span} < 1, x, \ldots, x^{n-2}, \cos(x), \sin(x) >, \quad x \in [0,\alpha],\ 0 < \alpha < \pi, \tag{8}$$

introduced and analyzed [(1)] in [4], but also the classical spaces

$$\text{span} < 1, x, \ldots, x^{n-2}, \mathrm{e}^{\rho \mathrm{x}}, \mathrm{e}^{-\rho \mathrm{x}} >,\ \rho \in \mathbb{R},$$

which generate exponential splines of arbitrary degree (see for example [7], [8] and references quoted therein).

Spaces of the form (4) with some additional restrictions on u, v have been also studied in [9], where generalized B-splines with simple knots are constructed and several properties of them established. However, in [9] it is explicitly assumed that $u^{(n-1)}, v^{(n-1)}$ are monotonic in $[a,b]$ and implicitly required (see [9] Lemma 4, Theorem 7) that $u^{(n-1)}, v^{(n-1)}$ only have one zero in the same interval. Thus, the class considered in [9] does not include the spaces (1) and (2) if $\alpha > \frac{\pi}{2}$, and (3) if $m_0, m_1 > 3$.

We mention that the results of Section 4 have been already obtained in [15] for spaces of the form span $< 1, x, u(x) >$, assuming u and its derivative strictly increasing and imposing additional requests on u related to symmetry.

[(1)] In [4] it is assumed that $0 \leq \alpha \leq \pi$. However, for $\alpha = \pi$ and $n = 2$ one of the elements of the Bernstein basis will vanish everywhere so therefore we did not consider this case.

Finally, blossoming and existence of Bernstein like bases have been nicely investigated in [5] (see also [12]) for a class of spaces which are not required to be Extended Tchebycheff spaces and which include spaces of the form (3). Hence, the results of our Section 4 can basically be found in [5]. However, here we present a different approach simply based on elementary analytic properties of the considered spaces. Thus, the simplicity of the proofs and of the formulation of the needed hypotheses make the mentioned section interesting.

The paper is divided into seven sections. In Section 2 we discuss zero properties of elements of the space $\mathcal{P}^n_{u,v}([a,b])$ and discuss Taylor- and Hermite interpolation. In the next section we analyze the approximation properties of the space $\mathcal{P}^n_{u,v}([a,b])$. The existence of a normalized B-basis for this space is discussed in Section 4 while in Section 5 we situate the space $\mathcal{P}^n_{u,v}([a,b])$ in the context of Tchebycheff systems. Section 6 is devoted to the presentation of an integral recurrence relation for the obtained B-basis and of the related degree raising process. Finally, in Section 7 we determine explicitly the coefficients for the degree-raising process for spaces of the form (4) with $u(x) = (1-x)^\mu$, $v(x) = x^\mu$, $\mu \in \mathbb{R}$, $\mu \geq n$. We also present a de Casteljau algorithm, and a geometric construction of B-splines for this space in the cubic case.

§2. Hermite interpolation

In this section we consider Hermite interpolation in the space $\mathcal{P}^n_{u,v}([a,b])$.

Lemma 1. *Suppose $\psi \in \mathcal{P}^n_{u,v}([a,b])$ has at least two distinct zeros in $[a,b]$ and at least $n+1$ zeros in $[a,b]$ counting multiplicities. If (6) holds then $\psi = 0$ on $[a,b]$.*

Proof: By Rolle's theorem the first derivative ψ' has at least two distinct zeros in $[a,b]$ and at least n zeros in $[a,b]$ counting multiplicities. Continuing we see that $\psi^{(n-1)}$ has at least two distinct zeros in $[a,b]$. Now $\psi(x) = p(x) + \gamma_u u(x) + \gamma_v v(x)$ for some polynomial p of degree $\leq n-2$ and some numbers γ_u and γ_v. It follows that $\psi^{(n-1)}(x) = \gamma_u u^{(n-1)}(x) + \gamma_v v^{(n-1)}(x)$. But (6) implies that $\gamma_u = \gamma_v = 0$ and then $p = 0$ since ψ has more than $n-2$ zeros. It follows that $\psi = 0$. □

We have a unique Hermite interpolant as long as we have at least two distinct interpolation points in $[a,b]$.

Lemma 2. *Let points $a \leq \tau_0 < \tau_1 < \cdots < \tau_r \leq b$ and positive integers $\mu_0, \mu_1, \ldots, \mu_r$ be given and let $f_{i,j} \in \mathbb{R}$, $i = 0,\ldots,r$, $j = 0,\ldots,\mu_i - 1$ be prescribed. Suppose also $r \geq 1$. Then there exists a unique $\psi \in \mathcal{P}^n_{u,v}([a,b])$, where $n+1 = \sum_{i=0}^r \mu_i$, such that*

$$\psi^{(j)}(\tau_i) = f_{i,j} \quad j = 0,\ldots,\mu_i - 1, \; i = 0,\ldots,r. \tag{9}$$

Proof: By Lemma 1 the homogeneous system corresponding to (9) has only the zero solution. □

We also have a unique interpolant at one point as long as this point belongs to the open interval (a,b). We call this interpolant the **Taylor polynomial** in $\mathcal{P}^n_{u,v}([a,b])$.

Lemma 3. *Let a point $c \in (a,b)$ and an integer $n \geq 2$ be given and let $f_j \in \mathbb{R}$, $j = 0, \dots, n$ be prescribed. If (7) holds there is a unique $\psi \in \mathcal{P}^n_{u,v}([a,b])$ such that*

$$\psi^{(j)}(c) = f_j \quad j = 0, \dots, n. \tag{10}$$

Proof: Let $\psi(x) = \sum_{i=0}^{n-2} \gamma_i x^i + \gamma_u u(x) + \gamma_v v(x)$ be a solution of the homogeneous system corresponding to (10). In particular $\psi^{(n-1)}(c) = \psi^{(n)}(c) = 0$ so from (7) it follows that $\psi^{(n-1)}(x) = 0$ for any $x \in [a,b]$. But then $\psi(x) = \sum_{i=0}^{n-2} \gamma_i x^i$ is a polynomial of degree $\leq n-2$ which has a zero of multiplicity $n+1$ at c. Thus $\psi(x) = 0$ on $[a,b]$. □

We end this section with some remarks about the conditions (6) and (7). Setting

$$\Delta^n_{u,v}(x_1, x_2) := \det \begin{pmatrix} u^{(n-1)}(x_1) & v^{(n-1)}(x_1) \\ u^{(n)}(x_2) & v^{(n)}(x_2) \end{pmatrix} \tag{11}$$

we notice that requiring

$$\Delta^n_{u,v}(x_1, x_2) \neq 0, \quad x_1, x_2 \in (a,b) \tag{12}$$

it is equivalent to say that if $\psi \in \mathcal{P}^n_{u,v}([a,b])$ and $\psi^{(n-1)}$ does not vanish everywhere in $[a,b]$, then either $\psi^{(n-1)}$ or $\psi^{(n)}$ is nonzero in (a,b). Thus (12) implies (6)-(7) while in general the converse is not true. This can be seen considering the spaces (1), (2). Indeed, these spaces verify (12) if $\alpha < \frac{\pi}{2}$ and (6)-(7) if $\alpha < \pi$.

§3. Approximation power

In this section we investigate in detail the Taylor expansion in the space $\mathcal{P}^n_{u,v}([a,b])$ and use it to discuss for the spaces (1),(2), and (3), the contribution of the function u, and v to the approximation power of the space $\mathcal{P}^n_{u,v}([a,b])$. In this section we only assume that (7) holds. This ensures that

$$\Delta^n_{u,v}(x, x) = \det \begin{pmatrix} u^{(n-1)}(x) & v^{(n-1)}(x) \\ u^{(n)}(x) & v^{(n)}(x) \end{pmatrix} \neq 0, \quad x \in (a,b). \tag{13}$$

We assume $[c,d]$ is a nontrivial subinterval of (a,b) and consider first the space $\mathcal{P}^n_{u,v}([c,d])$.

Lemma 4. *If (13) holds then the space $\mathcal{P}^n_{u,v}([c,d])$ is the nullspace of the differential operator $\mathrm{L}^n_{u,v} : C^{n+1}[c,d] \to C[c,d]$,*

$$\mathrm{L}^n_{u,v} := \mathrm{D}^{n+1} - \beta(x)\mathrm{D}^n - \gamma(x)\mathrm{D}^{n-1} \tag{14}$$

where $D^0 = I$, the identity,

$$\mathrm{D}^1 := \mathrm{D} := \frac{\mathrm{d}}{\mathrm{d}x}, \quad \mathrm{D}^n := \mathrm{D}\mathrm{D}^{n-1},$$

and

$$\begin{aligned} \beta(x) :=& \frac{u^{(n-1)}(x)v^{(n+1)}(x) - v^{(n-1)}(x)u^{(n+1)}(x)}{\Delta^n_{u,v}(x,x)}, \\ \gamma(x) :=& \frac{u^{(n+1)}(x)v^{(n)}(x) - v^{(n+1)}(x)u^{(n)}(x)}{\Delta^n_{u,v}(x,x)}. \end{aligned} \quad x \in [c,d]. \tag{15}$$

Proof: Clearly $\mathrm{L}^n_{u,v}(x^j) = 0, \;\; j = 0, \dots, n-2$. Moreover by a straightforward calculation

$$\begin{aligned} \Delta^n_{u,v}(x,x)\mathrm{L}(u(x)) =& [u^{(n-1)}(x)v^{(n)}(x) - u^{(n)}(x)v^{(n-1)}(x)]u^{(n+1)}(x) - \\ & -[u^{(n-1)}(x)v^{(n+1)}(x) - u^{(n+1)}(x)v^{(n-1)}(x)]u^{(n)}(x) - \\ & -[u^{(n+1)}(x)v^{(n)}(x) - u^{(n)}(x)v^{(n+1)}(x)]u^{(n-1)}(x) = 0. \end{aligned}$$

Similarly $\Delta^n_{u,v}(x,x)\mathrm{L}(v(x)) = 0$. □

Lemma 5. *If (13) holds then the Green's function associated with* $\mathrm{L}^n_{u,v}$ *is given by*

$$G^n_{u,v}(x,y) := \begin{cases} 0 & \text{if } c \le x \le y, \\ \phi(y)R_n(u,y)(x) + \delta(y)R_n(v,y)(x) & \text{if } y \le x \le d, \end{cases} \tag{16}$$

where

$$\phi(y) := -\frac{v^{(n-1)}(y)}{\Delta^n_{u,v}(y,y)}, \qquad \delta(y) := \frac{u^{(n-1)}(y)}{\Delta^n_{u,v}(y,y)}, \tag{17}$$

and

$$R_n(f,y)(x) := f(x) - \sum_{j=0}^{n-2} \frac{(x-y)^j}{j!} f^{(j)}(y) = \int_y^x \frac{(x-t)^{n-2}}{(n-2)!} f^{(n-1)}(t)dt.$$

Proof: Fix $y \in [c,d]$. We need to show that

$$\begin{aligned} G^n_{u,v}(x,y) =& 0, \quad x \in [c,y], \\ \mathrm{L}^n_{u,v}(G^n_{u,v}(x,y)) =& 0, \quad x \in [y,d], \\ \mathrm{D}^j(G^n_{u,v}(x,y))_{|x=y} =& \delta_{j,n}, \quad j = 0, \dots, n. \end{aligned} \tag{18}$$

The first two properties in (18) are obvious. Since $\mathrm{D}^j R_n(f,y)(x)_{|x=y} = 0$ for $j \le n-2$, we have $\mathrm{D}^j G^n_{u,v}(x,y)_{|x=y} = 0$ for $j \le n-2$. Moreover, from (17)

$$\begin{aligned} \mathrm{D}^{n-1}G^n_{u,v}(x,y)_{|x=y} &= \phi(y)u^{(n-1)}(y) + \delta(y)v^{(n-1)}(y) = 0, \\ \mathrm{D}^n G^n_{u,v}(x,y)(x)_{|x=y} &= \phi(y)u^{(n)}(y) + \delta(y)v^{(n)}(y) = 1. \qquad \square \end{aligned}$$

Since $G^n_{u,v}$ is the Green's function for the initial value problem

$$\begin{aligned} \mathrm{L}^n_{u,v}\psi(x) =& 0, \quad x \in (c,d), \\ \psi^{(j)}(c) =& f_j, \quad j = 0, \dots, n, \end{aligned}$$

we obtain the Taylor expansion with integral remainder in $\mathcal{P}^n_{u,v}([c,d])$ (cf. Theorems 10.7 and 10.8 in [16]).

Lemma 6. *Let $f \in C^{n+1}[c,d]$. If (13) holds then*

$$f(x) = \psi(x) + \int_c^d G^n_{u,v}(x,y)\mathrm{L}^n_{u,v} f(y)dy$$

where $\mathrm{L}^n_{u,v}$, $G^n_{u,v}$ are defined in Lemmas 4 and 5 and ψ is the unique element in $\mathcal{P}^n_{u,v}([a,b])$ such that

$$\psi^{(j)}(c) = f^{(j)}(c), \quad j = 0, \dots, n. \quad \square$$

In order to obtain a bound for the error term in the Taylor expansion we consider the following

Lemma 7. *If (13) holds then*

$$|G^n_{u,v}(x,y)| \le \frac{(x-y)^n}{n!} \frac{\max_{y \le z \le x} |\Delta^n_{u,v}(y,z)|}{|\Delta^n_{u,v}(y,y)|}, \quad c \le y < x \le d.$$

Proof: From Lemma 5 for $c \le y \le x \le d$

$$\begin{aligned} G^n_{u,v}(x,y) =& \phi(y) R_n(u,y)(x) + \delta(y) R_n(v,y)(x) \\ =& \int_y^x \frac{(x-t)^{n-2}[u^{(n-1)}(y)v^{(n-1)}(t) - v^{(n-1)}(y)u^{(n-1)}(t)]}{(n-2)!\Delta^n_{u,v}(y,y)} dt \\ =& \int_y^x \frac{(x-t)^{n-2}}{(n-2)!} \int_y^t \frac{\Delta^n_{u,v}(y,z)}{\Delta^n_{u,v}(y,y)} dz\, dt. \quad \square \end{aligned}$$

Finally, from the previous lemmas we immediately have the main result of this section

Theorem 8. *Assume that (13) holds. Let $f \in C^{n+1}[a,b]$ and $\psi \in \mathcal{P}^n_{u,v}([a,b])$ satisfy*

$$\psi^{(j)}(c) = f^{(j)}(c), \quad j = 0, \dots, n, \quad c \in (a,b),$$

Then for $c \le x < b$

$$|f(x) - \psi(x)| \le \max_{c \le y \le z \le x} \frac{|\Delta^n_{u,v}(y,z)|}{|\Delta^n_{u,v}(y,y)|} \frac{(x-c)^{n+1}}{(n+1)!} \|\mathrm{L}^n_{u,v} f\|_{\infty,[c,x]}. \quad \square$$

Example 9. *Consider the trigonometric case where $u(x) = \cos(x)$ and $v(x) = \sin(x)$ for $x \in [c,d]$ with $d - c < \pi$. From (11) it is easy to see that*

$$\Delta^n_{u,v}(y,z) = \cos(y-z), \quad c \le y \le z \le d \quad \text{and} \quad n \ge 2.$$

Thus for $[c,d] \subseteq [0,\alpha]$ with $\alpha < \pi$ we obtain from Theorem 8

$$|f(x) - \psi(x)| \le \frac{(x-c)^{n+1}}{(n+1)!} \|\mathrm{L}^n_{u,v} f\|_{\infty,[c,x]}, \quad c \le x \le \alpha,$$

where $\mathrm{L}^n_{u,v} = \mathrm{D}^{n-1}(\mathrm{D}^2 + 1)$. Thus letting $c \to 0$ and $d \to \alpha$ we find

$$|f(x) - \psi(x)| \le \frac{\alpha^{n+1}}{(n+1)!} \|\mathrm{D}^{n-1}(\mathrm{D}^2 + 1) f\|_{\infty,[0,\alpha]}, \quad 0 \le x \le \alpha.$$

Example 10. *Consider the case where* $u(x) = (1-x/h)^\mu$ *and* $v(x) = (x/h)^\mu$, $n \le \mu \in \mathbb{R}$, *on* $[a,b] = [0,h]$ *for some* $h > 0$ *and let* $0 < c < d < h$. *We find*

$$\frac{|\Delta^n_{u,v}(y,z)|}{|\Delta^n_{u,v}(y,y)|} = (1-y/h)\left(\frac{z}{y}\right)^{\mu-n} + (y/h)\left(\frac{1-z/h}{1-y/h}\right)^{\mu-n} =: g(z), \quad y \le z \le x.$$

By convexity g *achieves its maximum at one of the endpoints of the interval* $[y,x]$, *and since* $0 \le y/h \le 1$, *we obtain for* $c \le y \le z \le x \le d$

$$\begin{aligned} g(z) &\le \max\left\{1, \quad (1-y/h)\left(\frac{x}{y}\right)^{\mu-n} + \ (y/h)\left(\frac{1-x/h}{1-y/h}\right)^{\mu-n}\right\} \\ &\le \max\left\{1, \quad \left(\frac{x}{y}\right)^{\mu-n}, \left(\frac{1-x/h}{1-y/h}\right)^{\mu-n}\right\} \le \left(\frac{x}{y}\right)^{\mu-n} \le \left(\frac{x}{c}\right)^{\mu-n}. \end{aligned}$$

From Theorem 8 we obtain the upper bound

$$|f(x) - \psi(x)| \le \left(\frac{x}{c}\right)^{\mu-n} \frac{(x-c)^{n+1}}{(n+1)!} \|\mathrm{L}^n_{u,v} f\|_{\infty,[c,x]}, \quad 0 < c \le x \le d < h. \tag{19}$$

If $\mu = n$, *then* $\mathcal{P}^n_{u,v}([a,b])$ *reduces to the space of polynomials of degree* $\le n$, $\mathrm{L}^n_{u,v} = \mathrm{D}^{n+1}$, *and (19) gives the classical result for Taylor expansion in this space. On the other hand, if* $\mu > n$ *the bound (19) is not very good for small values of* c. *It turns out that functions like* u *and* v *in this example have nice shape preserving properties, but are not very useful for approximating smooth functions, at least not when* μ *is large. Indeed, it can be seen that the error in the best approximation of the constant 1 by the functions* $u^{(n-1)}$ *and* $v^{(n-1)}$ *on the interval* $[0,h]$ *is given by*

$$(2^{\mu-n} - 1)/(2^{\mu-n} + 1),$$

and this is very close to 1 when $\mu - n$ *is large. Thus, for large values of* $\mu - n$ *the approximation power of the system (4) for these* u *and* v *is essentially determined by how well the function* f *in question can be approximated by polynomials of degree* $\le n-2$. *On the contrary, if* $\mu - n \to 0$ *with a given rate, then the error in the best approximation of the constant 1 by the functions* $u^{(n-1)}$ *and* $v^{(n-1)}$ *on the interval* $[0,h]$ *approaches zero at the same rate.*

§4. Constructing a Normalized B-basis

In this section we construct a basis of $\mathcal{P}^n_{u,v}([a,b])$ which is normalized and totally positive and we prove that it is the normalized B-basis for this space. Since this basis has classical properties of the Bernstein basis for polynomials we will refer to it as the *Bernstein basis* for the space $\mathcal{P}^n_{u,v}([a,b])$. Throughout this section we only assume that condition (6) holds.

4.1 Constructing a Normalized Positive Basis

For each $l = 0, \dots, n-1$ it follows from Lemma 2 that the Hermite interpolation problem

$$\begin{aligned} \tilde{B}_l^{(i)}(a) &= 0, \quad i = 0, \dots, l-1, \\ \tilde{B}_l^{(l)}(a) &= 1, \\ \tilde{B}_l^{(i)}(b) &= 0, \quad i = 0, \dots, n-l-1, \end{aligned} \tag{20}$$

has a unique solution $\tilde{B}_l \in \mathcal{P}^n_{u,v}([a,b])$. Similarly for $l = n$ the Hermite interpolation problem

$$\begin{aligned} \tilde{B}_n^{(i)}(a) &= 0, \quad i = 0, \dots, n-1, \\ \tilde{B}_n(b) &= 1, \end{aligned} \tag{21}$$

has a unique solution $\tilde{B}_n \in \mathcal{P}^n_{u,v}([a,b])$. Since $\tilde{B}_l$ is continuous and positive near a for $l = 0, \dots, n-1$ and near b for $l = n$ it follows that

$$\tilde{B}_l(x) > 0, \quad x \in (a,b), \quad l = 0, \dots, n. \tag{22}$$

In fact, if there exists $x_l \in (a,b)$ such that $\tilde{B}_l(x_l) = 0$ then B_l has at least two distinct zeros in $[a,b]$ and at least $n+1$ zeros in $[a,b]$ counting multiplicities. By Lemma 1 $B_l = 0$, a contradiction.

It is immediate to verify that $\tilde{B}_l, l = 0, \dots, n$ are linearly independent so that the set

$$\{\tilde{B}_0, \dots, \tilde{B}_n\} \tag{23}$$

provides a basis for $\mathcal{P}^n_{u,v}([a,b])$. By a suitable scaling of the functions $\tilde{B}_l$ we obtain a positive partition of unity.

Theorem 11. *Let $\tilde{B}_l \in \mathcal{P}^n_{u,v}([a,b])$, $l = 0, \dots, n$, be constructed according to (20) and (21). Then there exist scalars b_l, $l = 0, \dots, n$ such that, setting*

$$B_l := b_l \tilde{B}_l \tag{24}$$

then

$$1 \equiv \sum_{l=0}^{n} B_l. \tag{25}$$

Moreover,

$$B_l(x) > 0, \quad x \in (a,b), \quad l = 0, \dots, n.$$

Proof: Since the constant function 1 belongs to the space $\mathcal{P}^n_{u,v}([a,b])$, it is possible to express it in a unique way as a linear combination of the elements of the basis (23). Thus, according to (22), it suffices to prove that it is possible to select $b_l > 0$, $l = 0, \dots, n$ such that (25) holds.

Since $\tilde{B}_l(a) = \delta_{0l}$ and $\tilde{B}_l(b) = \delta_{nl}$ we immediately obtain $b_0 = b_n = 1$ by evaluating (25) at $x = a$ and $x = b$. Fix $1 \le l \le n-1$. By (20) we have

$$\tilde{B}_r^{(l)}(a) = 0, \quad r = l+1, \dots, n.$$

Thus differentiating the equation $1=\sum_{r=0}^{n} b_r\tilde{B}_r(x)$, by (20) we find

$$0=\sum_{r=0}^{n} b_r\tilde{B}_r^{(l)}(a)=\sum_{r=0}^{l} b_r\tilde{B}_r^{(l)}(a)=\psi_l^{(l)}(a)+b_l,$$

where

$$\psi_l(x):=\sum_{r=0}^{l-1} b_r\tilde{B}_r(x).$$

Thus $b_l>0$ if and only if $\psi_l^{(l)}(a)<0$. Now by (20)

$$\begin{aligned}\psi_l(a)=&1\\ \psi_l^{(j)}(a)=&0, \qquad j=1,\ldots,l-1,\\ \psi_l^{(j)}(b)=&0, \qquad j=0,\ldots,n-l,\end{aligned}$$

so that ψ_l is a nonzero element of $\mathcal{P}_{u,v}^n([a,b])$ and it is not a constant. Moreover, $\psi_l^{(1)}$ has $n-1$ zeros at the endpoints of $[a,b]$. Hence, $\psi_l^{(n-1)}$ is not the zero function and, from Rolle's Theorem $\psi_l^{(l)}$ has at least $n-l$ zeros in $(a,b]$.

If $\psi_l^{(l)}(a)=0$ then $\psi_l^{(l)}$ has at least $n-l+1$ zeros in $[a,b]$ and at least two distinct zeros in $[a,b]$. By Rolle's Theorem this contradicts (6).

If $\psi_l^{(l)}(a)>0$ then, from the Taylor expansion at $x=a$, we have $\psi_l(x)>1$ if $x\in(a,b)$ and x is close enough to a. Then, there exists $x_l\in(a,b)$ such that $\psi_l(x_l)=1$. Thus, from Rolle's Theorem, there exists $x_l^{(1)}\in(a,x_l)$ such that $\psi_l^{(1)}(x_l^{(1)})=0$. Thus, $\psi_l^{(1)}$ has n zeros in $[a,b]$ and, by Rolle's Theorem, there exist $x_{1,l}\in(a,b),\ x_{2,l}\in[a,b]$, $x_{1,l}\neq x_{2,l}$, such that

$$\psi_l^{(n-1)}(x_{1,l})=0,\quad \psi_l^{(n-1)}(x_{2,l})=0,$$

which contradicts (6). $\square$

Remark 12. *The previous results ensure that, if the scalars b_l are selected in order to fulfill (25) then the set*

$$\{B_0,\ldots,B_n\} \tag{26}$$

is a normalized positive basis for $\mathcal{P}_{u,v}^n([a,b])$. In the following we will call this the Bernstein basis of $\mathcal{P}_{u,v}^n([a,b])$ and we will use sometimes the notation $B_{i,n}$ instead of $B_i,\ i=0,\ldots,n$.

4.2 Total Positivity

In this subsection we show that the Bernstein basis (26) is totally positive.

We recall that the basis (26) is (strictly) totally positive in a given interval I if any collocation matrix

$$M\begin{pmatrix} B_0\ , & \ldots, & B_n \\ t_0\ , & \ldots, & t_n \end{pmatrix} := \begin{pmatrix} B_0(t_0) & \ldots & B_n(t_0) \\ \vdots & & \vdots \\ B_0(t_n) & \ldots & B_n(t_n) \end{pmatrix} \tag{27}$$

with

$$t_0 < t_1 < \cdots < t_n, \quad t_i \in I,\ i = 0, \ldots, n$$

is (strictly) totally positive, that is all its sub-determinants are (positive) nonnegative.

As a first step we show the following lemma.

Lemma 13. *For $i = 0, \ldots, n$ and $l = 0, \ldots, n - i$, any nonzero element in*

$$\text{span} < B_i, B_{i+1}, \ldots, B_{i+l} >, \tag{28}$$

has at most l zeros in (a,b) including multiplicities.

Proof: Let $w_{i,l}$ be an element of (28) which has $l + 1$ zeros in (a, b). By (20) and (21) $w_{i,l}$ has a zero at a of multiplicity i and a zero at b of multiplicity $n - i - l$. This is a total of $n + 1$ zeros, Lemma 1 implies that $w_{i,l} = 0$ and we conclude that a nonzero element in (28) can have at most l zeros in (a, b) counting multiplicities. □

The following lemma will be useful for our main result.

Lemma 14. *Let $t_i < t_{i+1} < \cdots < t_{i+l}$ be given points in (a, b). For $i = 0, \ldots, n$ and $l = 0, \ldots, n - i$, there exists a unique*

$$z_{i,l} \in \text{span} < B_i, B_{i+1}, \ldots, B_{i+l} >,$$

such that

$$z_{i,l}(t_r) = \delta_{i,r},\ r = i, \ldots, i + l. \tag{29}$$

Moreover, $z_{i,l}(x) = \sum_{j=i}^{i+l} \zeta_j B_j(x)$ with

$$\zeta_i > 0. \tag{30}$$

Proof: From Lemma 13 we immediately have existence and uniqueness of $z_{i,l}$.

Now, let us examine the sign of ζ_i. Since $z_{i,l}$ has at least l zeros in (a, b) and is not the zero function, from Lemma 13

$$z_{i,l} \notin \text{span} < B_{i+1}, \ldots, B_{i+l} >$$

thus, $\zeta_i \neq 0$.

Since $B_n(x) > 0$ for any $x \in (a,b)$, we see that (30) holds for $i = n$. If $i < n$, assume that $\zeta_i < 0$. Then, from (20), (21) and (24)

$$z_{i,l}^{(j)}(a) = 0, \quad j = 0, \ldots, i-1$$
$$z_{i,l}^{(i)}(a) = \zeta_i b_i < 0.$$

Thus, from Taylor expansion at $x = a$, $z_{i,l}(x) < 0$ for $x \in (a,b)$ if x is close enough to a. Since $z_{i,l}(t_i) > 0$ there exists $x_{i,l} \in (a, t_i)$ such that $z_{i,l}(x_{i,l}) = 0$ and $z_{i,l}$ has at least $l+1$ zeros in (a,b). This contradicts Lemma 13. □

Now, we are ready to show the main result of this section

Theorem 15. *The basis (26) is strictly totally positive in (a,b) and totally positive in $[a,b]$.*

Proof: First we prove that any collocation matrix in (a,b) is strictly totally positive.

By a determinant identity on Page 8 in [6] (see also Theorem 2.5 in [1]) it is well known that a matrix is strictly totally positive provided that all minors with consecutive columns are positive.

Let $a < t_0 < \cdots < t_n < b$ be a sequence of collocation points.

For $i = 0, \ldots, n$ and $l = 0, \ldots, n-i$ let $M_{i,l}$ be the sub-matrix of order $l+1$ of the matrix (27) consisting of the rows and of the columns with indices $i+1, \ldots, i+l+1$ that is the sub-matrix corresponding to the basis elements

$$B_i, \ldots, B_{i+l}$$

evaluated at the collocation points

$$t_i, \ldots, t_{i+l}.$$

To prove that the basis (26) is strictly totally positive in (a,b) it suffices to show that $M_{i,l}$ has positive determinant. We will show this by induction on l.

If $l = 0$ the result follows from the positivity of B_i.

Now, assume that $M_{i+1,l-1}$ has positive determinant. Let $z_{i,l}$ be as in Lemma 14. Thus, its coefficients are the solution of the linear system

$$M_{i,l} \begin{pmatrix} \zeta_i \\ \zeta_{i+1} \\ \vdots \\ \zeta_{i+l} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So, from Cramer's rule,

$$\zeta_i = \frac{\det(M_{i+1,l-1})}{\det(M_{i,l})}.$$

The assertion follows from the induction hypothesis and from (30).

Total positivity of the basis (26) in $[a, b]$ follows from the strict total positivity by continuity arguments. □

Finally we remark that the Bernstein basis in $\mathcal{P}^n_{u,v}([a, b])$ is a B-basis in the sense of [14]. Recall that a totally positive basis $\{u_0, \ldots, u_n\}$ of a space $\mathcal{U}$ of continuous functions on an interval $[a, b]$ is a B-basis if for any other totally positive basis $\{v_0, \ldots, v_n\}$ of $\mathcal{U}$ the matrix K of change of basis

$$\{v_0, \ldots, v_n\} = K\{u_0, \ldots, u_n\}$$

is totally positive. It is shown in Theorem 3.2 of Chapter 4 in [14] that a totaly positive basis is a B-basis if for $0 \leq j < k \leq n$ the relations

$$\lim_{x \to a^+} \frac{u_k(x)}{u_j(x)} = \lim_{x \to b^-} \frac{u_j(x)}{u_k(x)} = 0$$

holds. By Theorem 15, (20) and (21) it follows that the Bernstein basis is the normalized B-basis for $\mathcal{P}^n_{u,v}([a, b])$.

Remark 16. *Whether hypothesis (6) is necessary and/or sufficient for the space (4) having a normalized B-basis depends both on n and on the choice of the functions u and v. For example, the space* span $< 1, \cos(x), \sin(x) >$ *admits a normalized B-basis only if x varies in an interval of length less than π (see [3] and Corollary 3.2 in [13]), so hypothesis (6) is necessary in order to ensure the existence of a normalized B-basis in this case. On the other hand, the space*

$$\text{span} < 1, x, , \ldots, x^{n-2}, \cos(x), \sin(x) >$$

admits a normalized B-basis on intervals of length greater than 2π if $n \geq 5$ (see [3]), so (6) is only sufficient for general values of n.

§5. Connection with Tchebycheff Systems

In this section we explicitly state some properties of the Bernstein basis for the space $\mathcal{P}^n_{u,v}([a, b])$ in connection with the theory of Tchebycheff systems. These properties are consequences of the results of the previous section.

For the sake of clarity we recall some definitions (see [16], Chapter 2).

Definition 17. *The Bernstein basis is a Tchebycheff (T)-system in I provided*

$$\det M \begin{pmatrix} B_0, & \ldots, & B_n \\ t_0\ , & \ldots, & t_n \end{pmatrix} > 0 \quad \text{for all } t_0 < t_1 < \cdots < t_n \in I.$$

Definition 18. *The Bernstein basis is a weak Tchebycheff (WT)-system in I provided*

$$\det M \begin{pmatrix} B_0, & \ldots, & B_n \\ t_0\ , & \cdots, & t_n \end{pmatrix} \geq 0 \quad \text{for all } t_0 < t_1 < \cdots < t_n \in I.$$

Definition 19. *The Bernstein basis is an Order Complete Tchebycheff (OCT)-system in I provided*

$$\{B_{i_0}, B_{i_1}, \dots, B_{i_k}\} \ \text{ is a T} - \text{system for all } i_0 < i_1 < \dots < i_k, \quad k = 0, \dots, n.$$

Definition 20. *The Bernstein basis is an Order Complete Weak Tchebycheff (OCWT)-system in I provided*

$$\{B_{i_0}, B_{i_1}, \dots, B_{i_k}\} \ \text{ is a WT} - \text{system for all } i_0 < i_1 < \dots < i_k, \ k = 0, \dots, n.$$

By strict total positivity we immediately have

Theorem 21. *The Bernstein basis is an (OCT)-system in (a, b) and an (OCWT)-system in $[a, b]$.*

Next let us consider the case of collocation matrices with derivatives. Let points

$$\tau_0 < \tau_1 < \dots < \tau_r \in I, \quad 0 \le r \le n$$

and integers

$$\nu_0, \nu_1, \dots, \nu_r$$

be given, with

$$\nu_i \ge 0, \ i = 0, \dots, r, \quad \sum_{i=0}^{r} \nu_i = n + 1.$$

We consider the points $t_0, \dots, t_n$ given by

$$t_0 = t_1 = \dots = t_{\nu_0 - 1} = \tau_0 < t_{\nu_0} = \dots = t_{\nu_0 + \nu_1 - 1} = \tau_1 < \dots < t_n = \tau_r \quad (31)$$

and define the matrix

$$\bar{M} \begin{pmatrix} B_0, & \cdots, & B_n \\ t_0\ , & \dots, & t_n \end{pmatrix} := \bar{M} \begin{pmatrix} B_0(\tau_0) & \cdots & B_n(\tau_0) \\ B_0^{(1)}(\tau_0) & \cdots & B_n^{(1)}(\tau_0) \\ \vdots & & \vdots \\ B_0^{(\nu_0 - 1)}(\tau_0) & \cdots & B_n^{(\nu_0 - 1)}(\tau_0) \\ B_0(\tau_1) & \cdots & B_n(\tau_1) \\ \vdots & & \vdots \\ B_0^{(\nu_r - 1)}(\tau_r) & \cdots & B_n^{(\nu_r - 1)}(\tau_r) \end{pmatrix}.$$

We recall the following definitions (see [16], Chapter 2)

Definition 22. *The Bernstein basis is an Extended Tchebycheff (ET)-system in I provided*

$$\det \bar{M} \begin{pmatrix} B_0, & \cdots, & B_n \\ t_0\ , & \cdots, & t_n \end{pmatrix} > 0 \ \text{ for all } t_0 \le t_1 \le \dots \le t_n \in I.$$

Definition 23. *The Bernstein basis is an Extended Weak Tchebycheff (EWT)-system in I provided*

$$\det \bar{M} \begin{pmatrix} B_0, & \cdots, & B_n \\ t_0\ , & \cdots, & t_n \end{pmatrix} \geq 0 \ \text{ for all } t_0 \leq t_1 \leq \cdots \leq t_n \in I.$$

Definition 24. *The Bernstein basis is an Order Complete Extended Tchebycheff (OCET)-system in I provided*

$$\{B_{i_0}, B_{i_1}, \ldots, B_{i_k}\} \ \text{ is an ET} - \text{system for all } i_0 < i_1 < \cdots < i_k, \ k = 0, \ldots, n.$$

Definition 25. *The Bernstein basis is an Order Complete Extended Weak Tchebycheff (OCEWT)-system in I provided*

$$\{B_{i_0}, B_{i_1}, \ldots, B_{i_k}\} \ \text{ is an EWT} - \text{system for all } i_0 < i_1 < \cdots < i_k, \ k = 0, \ldots, n.$$

Using the same arguments as in Lemma 14, the results of Theorem 15, and standard continuity arguments we have the following

Theorem 26. *The Bernstein basis is an OCET-system in (a, b) and, if (7) holds, an OCEWT system in $[a, b]$.*

§6. Additional properties of the Bernstein basis

In this section we briefly describe a recurrence integral relation to construct the elements of the basis (26) and a corresponding degree-raising algorithm. The results of this section are direct extensions of those presented in [4] for the space (8).

In this section we only assume that (6) holds.

6.1 An integral recurrence relation

From (6) we have that there exist unique elements, $U_{0,1,n}$, $U_{1,1,n}$ in span $< u^{(n-1)}(x), v^{n-1}(x) >$ such that

$$\begin{aligned} U_{0,1,n}(a) = 1,\ U_{0,1,n}(b) = 0, \\ U_{1,1,n}(a) = 0,\ U_{1,1,n}(b) = 1. \end{aligned} \tag{32}$$

Moreover, from (6),

$$U_{0,1,n}(x),\ U_{1,1,n}(x) > 0,\ x \in (a, b). \tag{33}$$

For $k = 2, \ldots, n$ let us define

$$\begin{aligned} U_{0,k,n}(x) &:= 1 - V_{0,k-1,n}(x), \\ U_{i,k,n}(x) &:= V_{i-1,k-1,n}(x) - V_{i,k-1,n}(x), \quad i = 1, \ldots, k-1 \\ U_{k,k,n}(x) &:= V_{k-1,k-1,n}(x), \end{aligned} \tag{34}$$

where for $i = 0, \ldots, k,\ k = 1, \ldots, n-1$

$$V_{i,k,n}(x) := \int_a^x U_{i,k,n}(t)dt/d_{i,k,n}$$

and

$$d_{i,k,n} := \int_a^b U_{i,k,n}(t)dt.$$

From, (32) and (34) by induction on k it can immediately be proved that, for $i = 0, \ldots, k$

$$\begin{aligned} U_{i,k,n}^{(j)}(a) &= 0, \quad j = 0, \ldots, i-1, \\ U_{i,k,n}^{(j)}(b) &= 0, \quad j = 0, \ldots, k-i-1. \end{aligned} \tag{35}$$

In addition, the functions

$$\{U_{0,k,n}, \ldots, U_{k,k,n}\} \tag{36}$$

belong to the space

$$\text{span} < 1, x, \ldots, x^{k-2}, u^{(n-k)}(x), v^{(n-k)}(x) >, \tag{37}$$

and $\sum_{i=0}^{k} U_{i,k,n}(x) \equiv 1$ for $k \geq 2$ so that, from (20), (21) and (25) we have that the set of functions (36) coincides with the Bernstein basis, (26), for the space (37). In particular, $U_{i,k,n}(x) > 0,\ x \in (a,b),\ i = 0, \ldots, k,\ k = 1, \ldots, n$, so that the recurrence relations (34) are well defined.

Moreover, for $k \geq 3$ we have

$$\frac{x-a}{b-a} = \sum_{i=0}^{k} \xi_{i,k,n} U_{i,k,n}(x)$$

for some numbers $\xi_{i,k,n}$. Evaluating at $x = a$ and $x = b$ and using (35) we see that $\xi_{0,k,n} = 0$ and $\xi_{k,k,n} = 1$ which together with (34) leads to

$$\frac{x-a}{b-a} = \sum_{i=0}^{k} V_{i,k-1,n}(x)\big(\xi_{i+1,k,n} - \xi_{i,k,n}\big),$$

where we assume $\xi_{k+1,k,n} = 0$. By differentiation

$$\frac{1}{b-a} = \sum_{i=0}^{k} U_{i,k-1,n}(x) \frac{\xi_{i+1,k,n} - \xi_{i,k,n}}{d_{i,k-1,n}}. \tag{38}$$

It follows that $\xi_{0,k,n} = 0$ and

$$\xi_{i+1,k,n} - \xi_{i,k,n} = \frac{d_{i,k-1,n}}{b-a}, \quad i = 0, 1, \ldots, k-1. \tag{39}$$

In particular, we obtain

$$\xi_{0,k,n} = 0,\ \xi_{1,k,n} = \frac{d_{0,k-1,n}}{b-a},\ \xi_{k-1,k,n} = 1 - \frac{d_{k-1,k-1,n}}{b-a},\ \xi_{k,k,n} = 1.$$

Summarizing, (34) provide a recurrence relation for constructing the Bernstein basis of $\mathcal{P}^n_{u,v}([a,b])$ and gives information about how to represent linear functions in terms of this basis.

Moreover, setting

$$\xi_{i,n} := (b-a)\xi_{i,n,n} + a,\ i = 0,\dots,n,$$

by (39) $a = \xi_{0,n} < \xi_{1,n} < \cdots < \xi_{n,n} = b$ and, for any $\psi = \sum_{i=0}^n b_i B_i$, the piecewise linear function connecting the points $(\xi_{0,n}, b_0), \dots, (\xi_{n,n}, b_n)$, has the geometric meaning of the classical control polygon.

6.2 Degree raising

In order to stress the dependence on the parameter n, in this subsection it is useful to use the notation $B_{i,n},\ i = 0,\dots,n$ for the Bernstein basis of $\mathcal{P}^n_{u,v}([a,b])$ defined in (26).

Obviously, $\mathcal{P}^n_{u,v}([a,b]) \subset \mathcal{P}^{n+1}_{u,v}([a,b])$ and a degree-raising algorithm exists, in the sense that any Bernstein basis function $B_{i,n}$ can be expressed as a linear combination of $B_{0,n+1},\dots,B_{n+1,n+1}$. We have the following result.

Theorem 27. *For $n \geq 2$ there exist a sequence $\{\theta_{0,n}, \theta_{1,n}, \dots, \theta_{n+1,n}\}$,*

$$\theta_{0,n} = 1,\ \theta_{n+1,n} = 0,\quad 0 \leq \theta_{i,n} \leq 1,\ i = 1,\dots,n,$$

such that

$$B_{i,n} = \theta_{i,n} B_{i,n+1} + (1-\theta_{i+1,n}) B_{i+1,n+1},\ i = 0,1,\dots,n. \tag{40}$$

Moreover,

$$\theta_{i,n} = \frac{B^{(i)}_{i,n}(a)}{B^{(i)}_{i,n+1}(a)} = \frac{d_{i-1,n,n+1} \cdots d_{0,n+1-i,n+1}}{d_{i-1,n-1,n} \cdots d_{0,n-i,n}},\ i = 1,\dots,n-1. \tag{41}$$

Proof: From Theorem 11 we know that, for $k = n, n+1$

$$\begin{aligned} B^{(i)}_{l,k}(a) =& 0,\ i = 0,\dots,l-1, \\ B^{(l)}_{l,k}(a) >& 0, \\ B^{(i)}_{l,k}(b) =& 0,\ i = 0,\dots,k-l-1, \end{aligned} \tag{42}$$

and therefore the relation between the bases of degree n and $n+1$ has the form

$$B_{i,n} = p_{i,n} B_{i,n+1} + q_{i+1,n} B_{i+1,n+1},$$

because additional terms would introduce extraneous non-zero derivatives at end points. We also know, from Subsection 4.1, that the basis functions must be positive for any $x \in (a, b)$ and therefore, again in virtue of (42), $p_{i,n}, q_{i+1,n} \geq 0$. Since

$$1 \equiv \sum_{i=0}^{n} B_{i,n} = \sum_{i=0}^{n} (p_{i,n} B_{i,n+1} + q_{i+1,n} B_{i+1,n+1}) = \sum_{i=0}^{n+1} B_{i,n+1} \ ,$$

we have $p_{0,n} = q_{n+1,n} = 1$ and $p_{i,n} + q_{i,n} = 1$ for $i = 1, \ldots, n$. Setting $\theta_{i,n} := p_{i,n}$ we have (40). From (34), (40) and (42) we obtain (41). □

§7. A Special Case

In this section we derive a degree raising formula for the Bernstein basis $B_{i,n}, \ i = 0, \ldots, n$ of the space

$$\begin{aligned} &\mathcal{P}^n_{u,v}([0,1]) = \text{span} < 1, x, \ldots, x^{n-2}, (1-x)^{\mu}, x^{\mu} > \ , \\ &\mu \in \mathbb{R}, \ \mu \geq n \ ; \ x \in [0,1], \ 2 \leq n \in \mathbb{N} \ , \end{aligned} \tag{43}$$

The Bernstein basis of $\mathcal{P}^n_{u,v}([a,b])$ is defined in (26). From Theorem 11,

$$B_{0,n}(x) = (1-x)^{\mu}, \ B_{n,n}(x) = x^{\mu}.$$

By the remark after (37), $U_{0,n-1,n}(x) = (1-x)^{\mu-1}$. Therefore, by (39)

$$\xi_{1,n} = \int_0^1 U_{0,n-1,n}(t)dt = \int_0^1 (1-t)^{\mu-1} dt = \frac{1}{\mu}. \tag{44}$$

Similarly,

$$\xi_{n-1,n} = 1 - \frac{1}{\mu}. \tag{45}$$

Consider now formula (40). If $\mu \geq n+1$, then $\theta_1 = 1, \ \theta_n = 0$. A more precise characterization for the θ_i is given in the following theorem.

Theorem 28. *If $\mu \geq n+1$, for the degree-raising process (40) we have the parameters*

$$\theta_{0,n} = 1 \ ; \ \theta_{i,n} = \frac{n-i}{n-1} \ , \ i = 1, \ldots, n \ ; \ \theta_{n+1,n} = 0 \ . \tag{46}$$

Proof: In the case $\mu \in \mathbb{N}$, the following explicit formula for the elements of the basis (26) of $\mathcal{P}^n_{u,v}([0,1])$ is provided in Corollary 4.2 in [5], in terms of the classical Bernstein basis, $\{\mathcal{B}_{0,\mu}, \ldots, \mathcal{B}_{\mu,\mu}\}$, of the space of polynomials of degree less than or equal to μ :

$$\begin{aligned} B_{0,n}(x) &= \mathcal{B}_{0,\mu}(x) = (1-x)^{\mu}, \ B_{n,n}(x) = \mathcal{B}_{\mu,\mu}(x) = x^{\mu} \\ B_{i,n}(x) &= \sum_{l=i}^{i+\mu-n} \alpha_{l,i,n} \mathcal{B}_{l,\mu}(x), \ i = 1, \ldots, n-1, \end{aligned} \tag{47}$$

where

$$\alpha_{l,j,k} = \frac{\binom{l-1}{j-1}\binom{\mu-l-1}{k-j-1}}{\binom{\mu-2}{k-2}} \; j = 1, \ldots, k-1, \tag{48}$$

if $l \geq j$, $\mu - l \geq k - j$, $k \geq j + 1 \geq 2$, and $\alpha_{l,j,k} = 0$ otherwise.

Now

$$\begin{aligned} & B_{i,n} - \theta_{i,n} B_{i,n+1} - (1 - \theta_{i+1,n}) B_{i+1,n+1} \\ &= B_{i,n} - \frac{n-i}{n-1} B_{i,n+1} - \frac{i}{n-1} B_{i+1,n+1} \\ &= \sum_{l=i}^{i+\mu-n} \big(\alpha_{l,i,n} - \frac{n-i}{n-1}\alpha_{l,i,n+1} - \frac{i}{n-1}\alpha_{l,i+1,n+1}\big)\mathcal{B}_{l,\mu} = 0. \end{aligned}$$

To show the last equality we note that since $k\binom{m}{k} = (m-k+1)\binom{m}{k-1}$ for $1 \leq k \leq m$ we have $\frac{n-i}{n-1}\alpha_{l,i,n+1} = \frac{\mu-l-n+i}{\mu-n}\alpha_{l,i,n}$ and $\frac{i}{n-1}\alpha_{l,i+1,n+1} = \frac{l-i}{\mu-n}\alpha_{l,i,n}$ and hence each coefficient of $\mathcal{B}_{l,\mu}$ in the sum vanishes.

In the general case $\mu \in \mathbb{R}$ we observe that from (41) and from the characterization of the elements of the basis (26) the function

$$f(\mu) := \theta_{i,n} - \frac{n-i}{n-1}, \; \mu \geq n+1$$

is a rational function of μ which vanishes at any $\mu \in \mathbb{N}, \mu \geq n+1$. So $f(\mu) \equiv 0$, that is (46) holds. □

We note that the values $\theta_{1,n}, \ldots, \theta_{n,n}$ given by (46) are exactly those given by the usual degree elevation process of the classical Bernstein polynomial of degree $n-2$. Since

$$\mathcal{P}^n_{u,v}([0,1]) = \text{span} < \mathcal{B}_{0,n-2}, \ldots, \mathcal{B}_{n-2,n-2}, (1-x)^\mu, x^\mu >,$$

where $\mathcal{B}_{i,n-2}$ denotes the ordinary Bernstein polynomials of degree $n-2$, an intuitive justification of (46) relies on the fact that (40) can be obtained applying the classical degree raising process to the polynomial part of $B_{i,n}$.

Now, let

$$x = \sum_{i=0}^{n} \xi_{i,n} B_{i,n}(x) \; ;$$

where, by (39), $\xi_{i+1,n} > \xi_{i,n}$, $i = 0, \ldots, n-1$ and (see (44), (45))

$$\xi_{0,n} = 0, \; \xi_{1,n} = \frac{1}{\mu}, \; \xi_{n-1,n} = 1 - \frac{1}{\mu}, \; \xi_{n,n} = 1. \tag{49}$$

A trivial consequence of (40) and (46) is that if $x = \sum_{i=0}^{n+1} \xi_{i,n+1} B_{i,n+1}(x)$, then, starting with $n = 3$ in (49),

$$\begin{aligned} & \xi_{0,n+1} = \xi_{0,n} \;, \; \xi_{n+1,n+1} = \xi_{n,n} \;; \\ & \xi_{i,n+1} = \frac{i-1}{n-1}\xi_{i-1,n} + \frac{n-i}{n-1}\xi_{i,n} \;, \; i = 1, \ldots, n. \end{aligned}$$

Fig. 1 shows some examples of the basis functions and their control polygon (see Subsection 6.1).

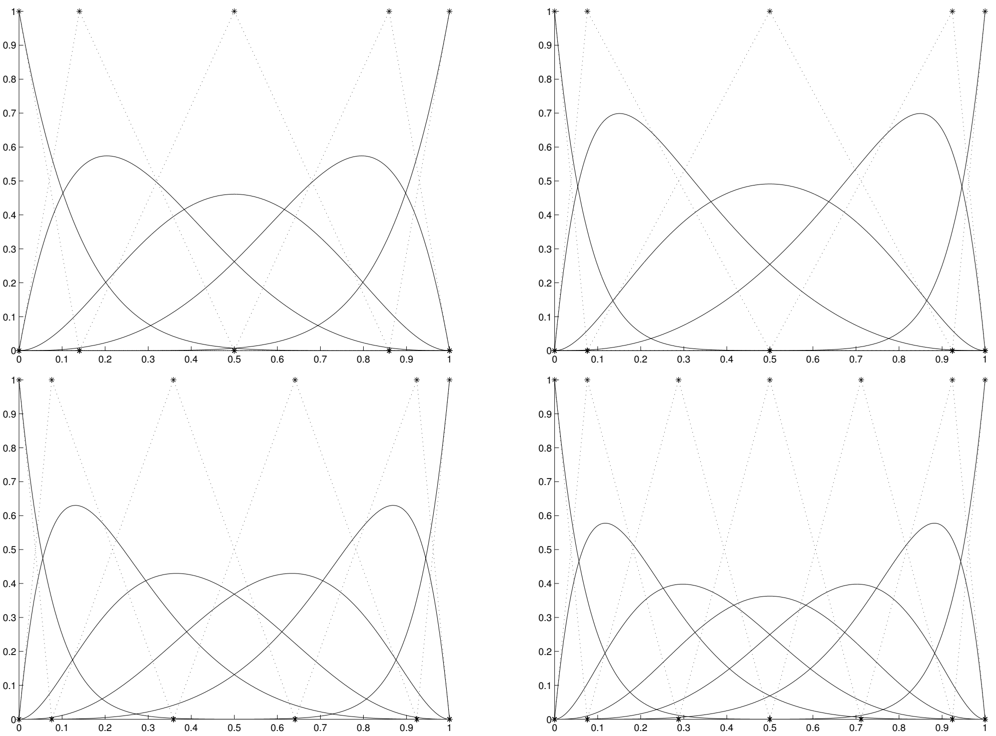

Fig. 1 The Bernstein bases and their control polygons for (in the clockwise sense): $n = 4, \mu = 7.13$; $n = 4, \mu = 10 + \pi$; $n = 5, \mu = 10 + \pi$; $n = 6, \mu = 10 + \pi$.

7.1 The cubic space

Let us consider the space

$$\mathcal{P}^3_{u,v}([0, 1]) = \text{span} < 1, x, (1 - x)^\mu, x^\mu > \ , \ x \in [0, 1], \quad \mu \geq 3. \tag{50}$$

In this case it is possible to explicitly compute the Bernstein basis in the form

$$\begin{pmatrix} B_{0,3} \\ B_{1,3} \\ B_{2,3} \\ B_{3,3} \end{pmatrix} = T^B_C \begin{pmatrix} 1 - x \\ x \\ (1 - x)^\mu \\ x^\mu \end{pmatrix},$$

where T^B_C is the transformation matrix which maps the "canonical" basis into the Bernstein one. Instead of repeating the process described in Theorem 11 we prefer to use Lemma 2 and use the equation $T^B_C = T^B_H \cdot T^H_C$, where T^B_H and T^H_C denote, respectively, the transformation matrices from the canonical to Hermite and from Hermite to Bernstein bases. The following theorem is obtained by straightforward computations, which are omitted for the sake of brevity.

Theorem 29. *We have*

$$\begin{pmatrix} B_{0,3} \\ B_{1,3} \\ B_{2,3} \\ B_{3,3} \end{pmatrix} = T^B_C \begin{pmatrix} 1 - x \\ x \\ (1 - x)^\mu \\ x^\mu \end{pmatrix},$$

where

$$T_C^B = \frac{1}{\mu(\mu-2)} \begin{pmatrix} 0 & 0 & \mu(\mu-2) & 0 \\ \mu(\mu-1) & -\mu & -\mu(\mu-1) & \mu \\ -\mu & \mu(\mu-1) & \mu & -\mu(\mu-1) \\ 0 & 0 & 0 & \mu(\mu-2) \end{pmatrix}. \quad \square$$

It is well known that for the cubic polynomial case given by $\mu = 3$ we can evaluate $\psi(x)$ using the de Casteljau algorithm, which, in matrix form, can be expressed as

$$\psi(x) = \begin{pmatrix} 1-x & x \end{pmatrix} \begin{pmatrix} 1-x & x & 0 \\ 0 & 1-x & x \end{pmatrix} \begin{pmatrix} 1-x & x & 0 & 0 \\ 0 & 1-x & x & 0 \\ 0 & 0 & 1-x & x \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

For the general "cubic" case it is simple to check the following result.

Theorem 30. *Let $\mu \in \mathbb{R}$, $\mu \geq 3$ and let $\psi = \sum_{j=0}^{3} b_j B_{j,3} \in \mathcal{P}^3_{u,v}([0,1])$ as in (50). Then*

$$\psi(x) = \begin{pmatrix} 1-x & x \end{pmatrix} \begin{pmatrix} 1-x & x & 0 \\ 0 & 1-x & x \end{pmatrix} M_\mu \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{51}$$

where, setting $\lambda := \mu - 2$,

$$M_\mu := \begin{pmatrix} (1-x)^\lambda & \frac{\lambda+1}{\lambda}\left(1-(1-x)^\lambda\right) - x & -\frac{1}{\lambda}\left(1-(1-x)^\lambda\right) + x & 0 \\ 0 & 1-x & x & 0 \\ 0 & -\frac{1}{\lambda}\left(1-x^\lambda\right) + (1-x) & \frac{\lambda+1}{\lambda}\left(1-x^\lambda\right) - (1-x) & x^\lambda \end{pmatrix}. \tag{52}$$ $\square$

From (51) we obtain a recursive de Casteljau like algorithm for evaluating $\psi \in \mathcal{P}^3_{u,v}([0,1])$.

By using a computer algebra system we easily obtain the following result.

Theorem 31. *The matrix M_μ defined in (52) is stochastic and totally positive.* $\square$

The previous theorem ensures that each step in the de Casteljau like algorithm involves only convex combinations of previous quantities. Thus, ψ inherits the geometric properties of its control polygon.

The question that naturally arises is whether it is possible to develop a "cubic" spline theory, similarly to that developed in [2] for the case of integer exponents. More precisely, given a sequence of extended knots

$$x_{-3} < x_{-2} < \ldots < x_{n+2} < x_{n+3},$$

and a sequence of real numbers

$$\boldsymbol{\mu} = \{\mu_{-3}, \mu_{-2}, \ldots, \mu_{n+2}\}, \ \mu_i \geq 3, \ i = -3, \ldots, n+2,$$

we set $h_i := x_{i+1} - x_i,\ i = -3, \cdots, n+2$ and we consider, for each interval $[x_i, x_{i+1}]$, the pair of functions (see (4))

$$u_i(t) := (1-t)^{\mu_i}, \quad v_i(t) = t^{\mu_i}, \quad t := (x - x_i)/h_i.$$

We wish to study the generalized cubic spline space

$$\begin{aligned}\mathcal{S}^3_{\boldsymbol{\mu}} := \{ s \in C^2[x_0, x_n] \text{ s.t. for } x \in [x_i, x_{i+1}],\ i = 0, \ldots, n-1 \\ s(x) = \psi_i(t);\ \psi_i \in \mathcal{P}^3_{u_i,v_i}([0,1]);\ t = (x - x_i)/h_i \}.\end{aligned}$$

Obviously, any piece ψ_i can be expressed as

$$\psi_i(x) = \sum_{j=0}^{3} b_{i,j} B_{j,3}((x - x_i)/h_i). \tag{53}$$

It turns out that the Bézier coefficients, $b_{i,j}$, can be computed using the analogous of the "geometric construction" (see, e.g., [10] for the classical cubic case and [2] for integer exponents), applying a repeated corner cutting process to the polygonal line connecting the B-splines coefficients $d_{-1}, \ldots, d_{n+1}$. Formally we have the following result.

Theorem 32. *The Bézier coefficients $b_{i,0}, b_{i,1}, b_{i,2}, b_{i,3}$ of the ith piece (53) depend only on the four B-spline coefficients $d_{i-1}, d_i, d_{i+1}, d_{i+2}$. They are given by*

$$\begin{aligned} &b_{i,1} = (1-\zeta_i) d_i + \zeta_i d_{i+1} \ ;\ b_{i,2} = \eta_i d_i + (1-\eta_i) d_{i+1} \ ; \\ &b_{i,0} = b_{i-1,3} = \omega_i b_{i-1,2} + (1-\omega_i) b_{i,1} \ , \end{aligned} \tag{54}$$

where

$$\zeta_i := \frac{\rho_i}{\rho_i + \tau_i + \sigma_i} \ ;\ \eta_i := \frac{\sigma_i}{\rho_i + \tau_i + \sigma_i} \ ;\ \omega_i := \frac{h_i/\mu_i}{h_{i-1}/\mu_{i-1} + h_i/\mu_i} \ .$$

and

$$\rho_i := \frac{\left(\frac{h_{i-1}}{\mu_{i-1}} + \frac{h_i}{\mu_i}\right)\frac{h_{i-1}}{\mu_{i-1}-1}}{\frac{h_{i-1}}{\mu_{i-1}-1} + \frac{h_i}{\mu_i - 1}} \ ;\ \sigma_i := \frac{\left(\frac{h_i}{\mu_i} + \frac{h_{i+1}}{\mu_{i+1}}\right)\frac{h_{i+1}}{\mu_{i+1}-1}}{\frac{h_i}{\mu_i-1} + \frac{h_{i+1}}{\mu_{i+1}-1}} \ ;\ \tau_i := (h_i - 2\frac{h_i}{\mu_i}). \quad \square$$

The proof simply consists of straightforward analytic computations which show that ψ_{i-1} and ψ_i, with Bézier coefficients given by (54), agree up to the second derivative at the knot x_i.

Following the approach of [2], Theorem 1, we obtain a basis of B-splines for the space $\mathcal{S}^3_{\boldsymbol{\mu}}$.

Theorem 33. *For $i = -1, 0, \ldots, n+1$ let N_i be given by (53) with Bézier coefficients computed according to (54) with $d_j = \delta_{i,j};\ j = -1, \ldots, n+1$. Then*

a) $N_i(x) > 0$ *for* $x \in (x_{i-2}, x_{i+2})$*;*
b) $N_i(x) = 0$ *for* $x \notin (x_{i-2}, x_{i+2})$*;*
c) $\sum_{i=-1}^{n+1} N_i(x) = 1$ *for* $x \in [x_0, x_n]$*;*
d) $N_i(x) \in \mathcal{S}^3_{\boldsymbol{\mu}}$. $\square$

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