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THE SENSITIVITY OF A SPLINE FUNCTION TO PERTURBATIONS OF THE KNOTS *

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Abstract.

In this paper we study the sensitivity of a spline function, represented in terms of B-splines, to perturbations of the knots. We do this by bounding the difference between a primary spline, and a secondary spline with the same B-spline coefficients, but different knots. We give a number of bounds for this difference, both local bounds and global bounds in general L^p-spaces. All the bounds are based on a simple identity for divided differences.

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1 Introduction.

All floating point computations are infested with round-off error, and computations with spline functions are no exception. To limit the round-off error in spline computations it is important to choose a basis with the property that small perturbations of the coefficients lead to a small perturbation of the spline represented by the coefficients. The B-spline basis is widely recognized as satisfying this criterion, in short it is said to be well conditioned. In this paper we investigate the conditioning of the B-spline basis from a different point of view, namely its sensitivity to perturbations of the knots. Simply stated, the conclusion is that the B-spline basis is also well conditioned with respect to the knots, under mild conditions on the perturbations.

The motivation for this investigation is practical. In some constructions of splines the knots result from floating point computations, and because of round-off error the separation between some of the knots can be very small. Since some spline computations can be troublesome when the knot spacing is highly nonuniform it is then tempting to let knots that are very close coalesce, or move

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knots so the spacing is more uniform. If these changes to the knot vector are small it is also tempting to leave the coefficients unchanged. In this paper we estimate the error that results from such actions. If one chooses to also adjust the coefficients, the combined perturbation error is estimated in Theorem 5.5.

This is closely related to questions concerning the continuity of a spline and its derivatives as functions of the knots, and our work elaborates on comments in [1] and Theorems 4.26 and 4.27 in [4]. We show that a continuous spline and its derivatives depend continuously on the knots in any L^p-space, while a discontinuous spline is only Lipschitz continuous with exponent $1/p$ in L^p.

Our results should also be useful for estimating the sensitivity of spline approximations to perturbations of the data.

In the next section we establish our notation and review some background material about splines. All our estimates are based on a simple identity for divided differences which corresponds to a similar identity for B-splines. These foundation results can be found in Section 3. In Section 4 we give estimates of the perturbation error in L^∞, and a local result that estimates the error at a specific point. Estimates in general L^p-spaces are given in Section 5. In Section 6, we shift our viewpoint slightly and discuss the continuity of a spline with respect to the knots. Finally, in Section 7, we briefly discuss generalizations to parametric curves.

2 Some background material

We will represent splines in terms of B-splines, see [4], and the polynomial order (degree + 1) will be denoted by k. Throughout the paper we will make use of two knot vectors $\boldsymbol{t} = (t_i)_{i=1}^{n+k}$ and $\boldsymbol{s} = (s_i)_{i=1}^{n+k}$, where $n \geq k$ is an integer. The knot vectors are nondecreasing sequences of real numbers which we call knots, with common k-tuple knots at each end, i.e.,

$$\begin{aligned} a = t_1 = \cdots = t_k = s_1 = \cdots = s_k \\ b = t_{n+1} = \cdots = t_{n+k} = s_{n+1} = \cdots = s_{n+k}. \end{aligned} \tag{2.1}$$

We think of $\boldsymbol{s}$ as a perturbation of $\boldsymbol{t}$. Corresponding to the knot vectors $\boldsymbol{t}$ and $\boldsymbol{s}$ we have the spline spaces

$$\mathbb{S}_{k,\boldsymbol{t}} = \Big\{ \sum_{i=1}^{n} c_i B_{i,k,\boldsymbol{t}} \mid c_i \in \mathbb{R} \Big\}, \qquad \mathbb{S}_{k,\boldsymbol{s}} = \Big\{ \sum_{i=1}^{n} c_i B_{i,k,\boldsymbol{s}} \mid c_i \in \mathbb{R} \Big\},$$

where the B's are B-splines of order k normalized to sum to one.

In our estimates, various quantities measuring the spacing of the knot vectors will occur. Some of these are

$$\left.\begin{aligned} \Delta_{i,r} &= t_{i+r} - t_i, \quad \text{for } i = 1, \ldots, n, \\ \underline{\Delta}_r &= \min_{1 \leq i \leq n} \{\Delta_{i,r} \mid \Delta_{i,r} > 0\}, \\ \overline{\Delta}_r &= \max_{1 \leq i \leq n} \Delta_{i,r}. \end{aligned}\right\} \quad \text{for } r = 1, \ldots, k. \tag{2.2}$$

We will also need some more complicated quantities which involve both knot vectors $\boldsymbol{t}$ and $\boldsymbol{s}$, see the end of Section 3 and Section 5.

We will make use of the p-norms for functions and vectors defined by

$$||f||_{L^p[a,b]} = ||f||_p = \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}, \qquad ||\boldsymbol{c}||_p = \left(\sum_{i=1}^n |c_i|^p\right)^{1/p} \tag{2.3}$$

for $p \geq 1$ and

$$||f||_{L^\infty[a,b]} = ||f||_\infty = \max_{a \leq x \leq b} |f(x)|, \qquad ||\boldsymbol{c}||_\infty = \max_{1 \leq i \leq n} |c_i|, \tag{2.4}$$

for $p = \infty$. In addition, for a spline $f = \sum_{i=1}^n c_i B_{i,k,\boldsymbol{t}}$ in $\mathbb{S}_{k,\boldsymbol{t}}$, we need the norm defined by

$$||\boldsymbol{c}||_{p,k,\boldsymbol{t}} = \begin{cases} \left(\sum_{i=1}^n |c_i|^p \Delta_{i,k}/k\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ ||\boldsymbol{c}||_\infty, & \text{if } p = \infty. \end{cases} \tag{2.5}$$

By a classical result due to de Boor, see [1], the size of a spline is bounded above and below by its B-spline coefficients,

$$\kappa_{k,p}^{-1} ||\boldsymbol{c}||_{p,k,\boldsymbol{t}} \leq ||f||_p \leq ||\boldsymbol{c}||_{p,k,\boldsymbol{t}}. \tag{2.6}$$

The norm of a B-spline can be bounded by setting $c_i = 1$ and $c_j = 0$ for $j \neq i$ in (2.6),

$$\frac{\Delta_{i,k}^{1/p}}{k^{1/p} \kappa_{k,p}} \leq ||B_{i,k,\boldsymbol{t}}||_p \leq \frac{\Delta_{i,k}^{1/p}}{k^{1/p}}. \tag{2.7}$$

Another consequence of (2.6) is that

$$||\boldsymbol{c}||_p \leq \left(\frac{k}{\underline{\Delta}_k}\right)^{1/p} ||\boldsymbol{c}||_{p,k,t} \leq \kappa_{k,p} \left(\frac{k}{\underline{\Delta}_k}\right)^{1/p} ||f||_p, \quad 1 \leq p \leq \infty, \tag{2.8}$$

where $\underline{\Delta}_k$ is given by (2.2).

If $f = \sum_i c_i B_{i,k,\boldsymbol{t}}$ and $g = \sum_i d_i B_{i,k,\boldsymbol{t}}$ are two splines with B-spline coefficients $\boldsymbol{c}$ and $\boldsymbol{d}$ such that $f \neq 0$, then the inequalities (2.6) lead to

$$\frac{||f - g||_p}{||f||_p} \leq \kappa_{k,p} \frac{||c - d||_{p,k,\boldsymbol{t}}}{||c||_{p,k,\boldsymbol{t}}} \tag{2.9}$$

which explains why $\kappa_{k,p}$ is called the p-norm condition number of the B-spline basis.

The popularity of the B-spline basis may be partly attributed to the fact that it is well conditioned, at least for low degrees. It was recently shown by Scherer and Shadrin [3] that the condition number satisfies the bound

$$\kappa_{k,p} \leq k 2^k \tag{2.10}$$

for any p in the range $1 \leq p \leq \infty$.

3 Three lemmas

The primary basis for our work is the following simple property of divided differences.

LEMMA 3.1. *For any real numbers $t_1, \cdots, t_n$ and $s_1, \cdots, s_n$, and any sufficiently smooth function h we have*

$$[t_1, \cdots, t_n]h - [s_1, \cdots, s_n]h = \sum_{j=1}^{n}(t_j - s_j)[s_1, \cdots s_j, t_j, \cdots, t_n]h. \tag{3.1}$$

PROOF. By applying the recurrence relation for divided differences, we may write each term on the right-hand side as a difference of divided differences of one order lower. The right hand side then becomes

$$\sum_{j=1}^{n}\big([s_1, \cdots, s_{j-1}, t_j, \cdots, t_n]h - [s_1, \cdots, s_j, t_{j+1}, \cdots, t_n]h\big).$$

But this sum simplifies immediately to the left hand side. □

Since B-splines can be defined in terms of divided differences, the identity (3.1) translates into an identity for splines that will prove useful to us.

LEMMA 3.2. *Suppose $f \in \mathbb{S}_{k,\boldsymbol{t}}$ and $g \in \mathbb{S}_{k,\boldsymbol{s}}$ have the same B-spline coefficients $\boldsymbol{c} = (c_i)_{i=1}^n$. Then*

$$f(x) - g(x) = \sum_{t_j \neq s_j}(t_j - s_j)\sum_{i=j-k+1}^{j}(c_{i-1} - c_i)M_{i,j}(x), \tag{3.2}$$

where

$$M_{i,j}(x) = [s_i, \cdots s_j, t_j, \cdots, t_{i+k-1}](y - x)_+^{k-1}, \quad x \in \mathbb{R}, \tag{3.3}$$

is a B-spline of order k. Equivalently,

$$f(x) - g(x) = \sum_{i=2}^{n}(c_{i-1} - c_i)\sum_{j=i}^{i+k-1}(t_j - s_j)M_{i,j}(x). \tag{3.4}$$

PROOF. Recall that B-splines can be expressed in terms of divided differences as

$$B_{i,k,\boldsymbol{t}}(x) = (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](y - x)_+^{k-1}$$

where a_+ denotes a if a is positive and zero otherwise.

For fixed x let $h(y) = (y - x)_+^{k-1}$. By the recurrence relation for divided differences we have

$$\begin{aligned}\sum_{i=1}^{n} c_i B_{i,k,\boldsymbol{t}}(x) &= \sum_{i=1}^{n} c_i (t_{i+k} - t_i)[t_i, \cdots, t_{i+k}]h \\ &= \sum_{i=1}^{n} c_i ([t_{i+1}, \cdots, t_{i+k}]h - [t_i, \cdots, t_{i+k-1}]h) \\ &= \sum_{i=2}^{n} (c_{i-1} - c_i)[t_i, \cdots, t_{i+k-1}]h + c_n.\end{aligned}$$

The last equality follows by rearranging sums, and from the observations that $[t_1, \cdots, t_k]h = 0$ and $[t_{n+1}, \cdots, t_{n+k}]h = 1$ for $x \in [a, b]$. ¿From this it follows that

$$f(x) - g(x) = \sum_{i=2}^{n} (c_{i-1} - c_i)([t_i, \cdots, t_{i+k-1}]h - [s_i, \cdots, s_{i+k-1}]h).$$

Applying Lemma 3.1 to the right-hand side then leads to (3.4). Equation (3.2) follows from (3.4) by rearranging the sums and making use of the fact that $s_i = t_i$ for $i \le k$ and $i > n$. □

Each of the $M_{i,j}$ is an ordinary B-spline, although it is not normalized in the standard way. What is unusual is that two knot vectors are involved in its definition so that when i and j vary, a much larger family of B-splines are generated than the two bases $\{B_{i,k,\boldsymbol{t}}\}_{i=1}^{n}$ and $\{B_{i,k,\boldsymbol{s}}\}_{i=1}^{n}$. To obtain estimates of the local perturbation error we need to know how many of the $M_{i,j}$ that are nonzero at a given x. This is answered by the following lemma.

LEMMA 3.3. *Let $M_{i,j}$ be as in (3.3), suppose that x is in the interval $[t_k, t_{n+1})$, that the integers μ and ν are defined by*

$$x \in [t_\mu, t_{\mu+1}) \cap [s_\nu, s_{\nu+1}), \tag{3.5}$$

and that f and g have the same B-spline coefficients $\mathbf{c}$. Then

$$f(x) - g(x) = \sum_{i \in I(x)} (c_{i-1} - c_i) \sum_{j=i}^{i+k-1} (t_j - s_j) M_{i,j}(x), \tag{3.6}$$

where the index set $I(x)$ is given by

$$I(x) = \begin{cases} \{\mu - k + 2, \ldots, \nu\}, & \text{if } \nu \ge \mu; \\ \{\nu - k + 2, \ldots, \mu\}, & \text{if } \mu \ge \nu; \end{cases}$$

with $M_{i,j}(x)$ being zero for all other values of i and j. Equivalently

$$f(x) - g(x) = \sum_{j \in J(x)} (t_j - s_j) \sum_{i \in I(x) \cap \{j-k+1, \ldots, j\}} (c_{i-1} - c_i) M_{i,j}(x), \tag{3.7}$$

where

$$J(x) = \begin{cases} \{\mu - k + 2, \dots, \nu + k - 1\}, & \text{if } \nu \geq \mu; \\ \{\nu - k + 2, \dots, \mu + k - 1\}, & \text{if } \mu \geq \nu. \end{cases}$$

Thus

$$\Big| \big\{ (i,j) \mid M_{i,j}(x) \neq 0 \big\} \Big| \leq k(k + |\nu - \mu| - 1), \tag{3.8}$$

where $|S|$ *denotes the number of elements in the set* S.

Proof. We start by observing that μ and ν are uniquely defined by (3.5). Suppose that $\nu \geq \mu$. Recall that the knots of $M_{i,j}$ are

$$(s_i, \dots, s_j, t_j, \dots, t_{i+k-1}), \tag{3.9}$$

and j runs from i to $i + k - 1$ while i runs from 2 to n. If $i \leq \mu - k + 1$ then $t_{i+k-1} \leq t_\mu$ and $s_{i+k-1} \leq s_\mu \leq s_\nu$ so that all the knots in (3.9) are to the left of both t_μ and s_ν and hence $M_{i,j}(x) = 0$. On the other hand, if $i \geq \nu + 1$ then $t_i \geq t_{\nu+1} \geq t_{\mu+1}$ and $s_i \geq s_{\nu+1}$, so all the knots of $M_{i,j}$ are to the right of both $t_{\mu+1}$ and $s_{\nu+1}$ and therefore $M_{i,j}(x) = 0$. Hence equation (3.6) follows from (3.4). Changing the order of summation in (3.6) we obtain (3.7).

For the remaining $k + \nu - \mu - 1$ values of i each of the k B-splines resulting from different values of j have knots both to the right and left of the interval $[t_\mu, t_{\mu+1}) \cap [s_\nu, s_{\nu+1})$ so $M_{i,j}(x)$ can be nonzero and there are $k(k + \nu - \mu - 1)$ possibly nonzero B-splines on this interval, so (3.8) follows.

The case $\nu \leq \mu$ follows by the symmetry of $\boldsymbol{s}$ and $\boldsymbol{t}$. □

Since $\boldsymbol{t}$ and $\boldsymbol{s}$ agree at the ends, the restriction on the sums in (3.6) and (3.7) can be improved somewhat. For example we can improve (3.6) to

$$f(x) - g(x) = \sum_{i \in I(x)} (c_{i-1} - c_i) \sum_{j=\max(i,k+1)}^{\min(i+k-1,n)} (t_j - s_j) M_{i,j}(x).$$

However, in general this improvement is only marginal and would complicate our bounds.

These lemmas provide the basic tools for obtaining estimates of the perturbation error. But our results will depend on how the knots in $\boldsymbol{s}$ and $\boldsymbol{t}$ are interlaced so we need some quantities that measure this. Let i and j be fixed integers with $2 \leq i \leq n$ and $i \leq j \leq i + k - 1$ and consider the $k + 1$ knots

$$\tilde{\boldsymbol{u}} = (s_i, \dots, s_j, t_j, \dots, t_{i+k-1}).$$

Let $\boldsymbol{u} = (u_\ell)_{\ell=0}^k$ denote the knots in $\tilde{\boldsymbol{u}}$, listed in nondecreasing order. We define the quantities

$$\begin{aligned} \delta_{i,j,r} &= \min_{0 \leq \ell \leq k-r} \{u_{\ell+r} - u_\ell \mid u_{\ell+r} - u_\ell > 0\}, \\ \delta_{i,r} &= \min_{i \leq j \leq i+k-1} \{\delta_{i,j,r} \mid \delta_{i,j,r} > 0\}, \\ \delta_r &= \min_{2 \leq i \leq n} \{\delta_{i,r} \mid \delta_{i,r} > 0\}. \end{aligned} \tag{3.10}$$

Note that unless $s_i = s_j = t_j = t_{i+k-1}$ at least one $\delta_{i,j,r}$ will be positive so that $\delta_{i,r}$ and δ_r are then well defined. In the case where $s_i = s_j = t_j = t_{i+k-1}$ we will not be interested in $\delta_{i,j,r}$ so we leave it undefined.

It is a simple matter to check that $\delta_{i,j,k}$ can be expressed as

$$\delta_{i,j,k} = \max\{t_{i+k-1} - s_i, t_{i+k-1} - t_j, s_j - t_j, s_j - s_i\}. \tag{3.11}$$

For our estimates in later sections we need to bound the L^p-norm of the B-spline $M_{i,j}$. The required inequalities follow immediately from (2.7), and may, after appropriate scaling, be written as

$$\frac{1}{\kappa_{k,p}} \cdot \frac{1}{k^{1/p}\,\delta_{i,j,k}^{1-1/p}} \leq ||M_{i,j}||_p \leq \frac{1}{k^{1/p}\,\delta_{i,j,k}^{1-1/p}}. \tag{3.12}$$

4 Perturbation estimates in L^∞

The main results of this section are Theorems 4.1, 4.3 and 4.5. Theorem 4.1 provides a very simple upper bound on the perturbation error, with no restriction on the knots. Theorem 4.3 bounds the error in the special case where only one, possibly multiple, knot is perturbed, while Theorem 4.5 gives a bound that should be useful in a situation where many knots are perturbed.

For our purposes, identity (3.2) in Lemma 3.2 has the appealing feature that the difference $t_j - s_j$ between corresponding knots in the two knot vectors appears explicitly. This makes it simple to estimate the difference $f - g$ by applying the rightmost inequality in (3.12) (for $p = \infty$).

THEOREM 4.1. *If $f \in \mathbb{S}_{k,\boldsymbol{t}}$ and $g \in \mathbb{S}_{k,\boldsymbol{s}}$ have the same B-spline coefficients $\boldsymbol{c}$ then*

$$||f - g||_\infty \leq \sum_{t_j \neq s_j} |t_j - s_j| \sum_{i=j-k+1}^{j} |c_i - c_{i-1}|/\delta_{i,j,k}. \tag{4.1}$$

¿From (3.11) we have $\delta_{i,j,k} \geq |t_j - s_j|$, so inequality (4.1) is always valid and indicates that if $t_j - s_j$ is small for all j then f will be close to g in the L^∞-norm. But this is not the complete truth. The right-hand side of (4.1) is not small if $\delta_{i,j,k}$ is comparable to $t_j - s_j$ in magnitude for some i and j. From the expression (3.11) for $\delta_{i,j,k}$ we see that this happens if all the differences $t_{i+k-1} - s_i$, $t_{i+k-1} - t_j$ and $s_j - s_i$ are of the same order of magnitude as $s_j - t_j$. A typical situation in which this can happen is if $t_i = t_{i+k-1}$ for some i, see the end of Section 6.

The estimate (4.1) can be sharpened if all the perturbations have the same sign. (For simplicity we have assumed that $t_i \geq s_i$ for all i; a symmetric result does of course hold in the complementary situation.)

LEMMA 4.2. *If $f \in \mathbb{S}_{k,\boldsymbol{t}}$ and $g \in \mathbb{S}_{k,\boldsymbol{s}}$ have the same B-spline coefficients $\boldsymbol{c}$ and if $t_i \geq s_i$ for all i then*

$$||f - g||_\infty \leq \sum_{t_j \neq s_j} |t_j - s_j| \max_{j-k+1 \leq i \leq j} \frac{|c_i - c_{i-1}|}{t_{i+k-1} - s_i}. \tag{4.2}$$

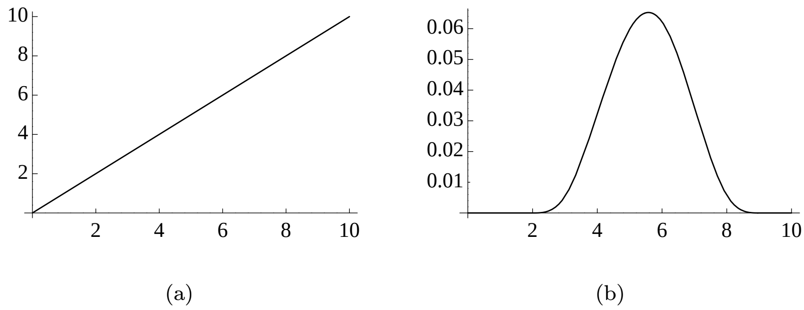

Figure 4.1: Perturbation of a linear spline

PROOF. We define normalized B-splines by

$$N_{i,j} = (t_{i+k-1} - s_i)M_{i,j}.$$

Since $s_i \le t_i$ for all i we see that for fixed j the $\{N_{i,j}\}_{i=j-k+1}^{j}$ are consecutive B-splines on the nondecreasing knots $s_{j-k+1}, \ldots, s_j, t_j, \ldots, t_{j+k-1}$. Moreover, since in this case $\delta_{i,j,k} = t_{i+k-1} - s_i$ is the support of $M_{i,j}$, we have from classical B-spline theory, see [4], the inequality

$$\sum_{i=j-k+1}^{j} N_{i,j}(x) \le 1, \quad \text{for} \quad x \in [a,b].$$

By writing (3.2) in the form

$$f(x) - g(x) = \sum_{t_j \ne s_j} (t_j - s_j) \sum_{i=j-k+1}^{j} \frac{c_{i-1} - c_i}{t_{i+k-1} - s_i} N_{i,j}(x) \tag{4.3}$$

the result now follows. □

The following example shows that the bound in (4.2) is sharp.

EXAMPLE 4.1. A study of the proof of the inequality in Lemma 4.2 shows that the worst case occurs when the difference between the B-spline coefficients is constant and the interior knots are uniform. In Figure 1 (a) we have represented the line $y = x$ on the interval $[0, 10]$ as a cubic spline $f(x)$ with knots

$$\boldsymbol{t} = (0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10).$$

In Figure 1 (b) we show $f(x) - g(x)$ where the cubic spline g differs from f only in that the knots at 5 and 6 have been moved to the locations 5.1 and 6.1 respectively. We see that $||f - g||_\infty$ is approximately equal to 0.07. This is in agreement with inequality (4.2). We have $k = 4$, $t_i = s_i$ for $i \ne 9, 10$, and $t_{i+k-1} - s_i \ge 2.9$ for $i = 6, 7, 8, 9, 10$. Moreover $c_i - c_{i-1} = 1$ for $i = 6, 7, 8, 9$.

Therefore the right-hand side of (4.2) becomes $(s_9 - t_9)/2.9 + (s_{10} - t_{10})/2.9 = 2/29 \approx 0.07$.

We consider next the perturbation of one (possibly multiple) knot.

THEOREM 4.3. *Suppose for some integers* j, r *with* $k+1 \le j \le n$ *and* $r \ge 1$ *that* $t_{j-1} < t_j = \cdots = t_{j+r-1} < t_{j+r}$. *If* $\boldsymbol{s}$ *is such that* $s_i = t_i$ *for* $i \notin \{j, \ldots, j+r-1\}$ *and* $t_{j-1} \le s_j = \cdots = s_{j+r-1} < t_j$, *and if* $f \in \mathbb{S}_{k,\boldsymbol{t}}$ *and* $g \in \mathbb{S}_{k,\boldsymbol{s}}$ *have the same B-spline coefficients* $\boldsymbol{c}$ *then*

$$||f - g||_\infty \le r|t_j - s_j| \max_{j-k+1 \le i \le j+r-1} \frac{|c_i - c_{i-1}|}{t_{i+k-1} - s_i}. \tag{4.4}$$

PROOF. This follows immediately from Lemma 4.2. □

Note that Theorem 4.3 subsumes (for $r = k - 1$) the special case of the (continuous) Bernstein/Bézier representation of splines.

By making use of Lemma 3.3 we can give a general, local result.

LEMMA 4.4. *Suppose* $f \in \mathbb{S}_{k,\boldsymbol{t}}$ *and* $g \in \mathbb{S}_{k,\boldsymbol{s}}$ *have the same B-spline coefficients* $\boldsymbol{c}$, *let* x *be a fixed number in the interval* $[t_k, t_{n+1})$, *and suppose that the integers* μ *and* ν *are defined by*

$$x \in [t_\mu, t_{\mu+1}) \cap [s_\nu, s_{\nu+1}). \tag{4.5}$$

Then

$$\big|f(x) - g(x)\big| \le k(k + |\nu - \mu| - 1) \max_{i \in I(x)} \frac{|c_{i-1} - c_i|}{\delta_{i,k}} \max_{j \in J(x)} |t_j - s_j|, \tag{4.6}$$

where the index sets $I(x)$ *and* $J(x)$ *are defined in Lemma 3.3.*

PROOF. ¿From Lemma 3.3 we have

$$f(x) - g(x) = \sum_{i \in I(x)} (c_{i-1} - c_i) \sum_{j=i}^{i+k-1} (t_j - s_j) M_{i,j}(x).$$

The result follows by bounding $M_{i,j}(x)$ as in (3.12) and making use of (3.8). □

The quantity $\max_x |\nu - \mu|$ will occur repeatedly in the rest of the paper and deserves a proper name.

DEFINITION 4.1. *For each* x *in* $[t_k, t_{n+1})$ *let* $\mu = \mu(x)$ *and* $\nu = \nu(x)$ *denote the integers defined by the relation* $x \in [t_\mu, t_{\mu+1}) \cap [s_\nu, s_{\nu+1})$. *The number*

$$L = L(\boldsymbol{t}, \boldsymbol{s}) = \max_{x \in [a,b)} |\nu(x) - \mu(x)| \tag{4.7}$$

is called the perturbation number *of* $\boldsymbol{s}$ *and* $\boldsymbol{t}$.

In practice, a knot is rarely perturbed beyond a neighbouring knot. With this restriction we have $L \le \max\{m_{\boldsymbol{s}}, m_{\boldsymbol{t}}\}$, where $m_{\boldsymbol{t}}$ is the maximum multiplicity of the interiour knots $t_{k+1}, \ldots, t_n$ of $\boldsymbol{t}$, and $m_{\boldsymbol{s}}$ is defined similarly.

Lemma 4.4 allows us to bound the L^∞-norm of the perturbation error in terms of the ℓ^∞-norm of $\boldsymbol{t} - \boldsymbol{s}$.

THEOREM 4.5. *If $f \in \mathbb{S}_{k,\boldsymbol{t}}$ and $g \in \mathbb{S}_{k,\boldsymbol{s}}$ have the same B-spline coefficients $\boldsymbol{c}$ then*

$$||f-g||_\infty \le \frac{(L+k-1)k}{\delta_k} ||\boldsymbol{t}-\boldsymbol{s}||_\infty \max_{2\le i\le n} |c_{i-1}-c_i|, \tag{4.8}$$

where $L = L(\boldsymbol{s},\boldsymbol{t})$ is the perturbation number of $\boldsymbol{s}$ and $\boldsymbol{t}$.

The inequalities (4.6) and (4.8) are not sharp since the bound on $M_{i,j}$ cannot in general be sharp for all values of i and j. To get a better estimate we need a good bound for the sum $\sum_{j=i}^{i+k-1} M_{i,j}(x)$, as in the special situation in Lemma 4.2, but this seems difficult. In any case, the bounds (4.6) and (4.8) have the advantage that they do not depend on the length of the knot vectors, provided the knots in $\boldsymbol{s}$ are not too different from the knots in $\boldsymbol{t}$.

5 Bounds in L^p

In this section and the next p will always indicate the L^p-space we are working in and q will denote the dual index given by

$$\frac{1}{p}+\frac{1}{q}=1.$$

This section also contains three main results. Theorem 5.3 gives an easily computable upper bound on the L^p-error, while Theorem 5.4 gives two bounds in terms of the given spline f and its derivative. Theorem 5.5 gives a bound where both B-spline coefficients and knots are perturbed.

Theorem 4.1 has a natural generalization to L^p, and the following theorem is obtained by taking L^p norms in (3.2)), using the triangle inequality, and applying (3.12)

THEOREM 5.1. *If $f \in \mathbb{S}_{k,\boldsymbol{t}}$ and $g \in \mathbb{S}_{k,\boldsymbol{s}}$ have the same B-spline coefficients $\boldsymbol{c}$ then for $1 \le p \le \infty$*

$$||f-g||_p \le k^{-1/p} \sum_{t_j\neq s_j} |t_j-s_j| \sum_{i=j-k+1}^{j} |c_i-c_{i-1}|/\delta_{i,j,k}^{1/q}.$$

Note that we obtain Theorem 4.1 as a special case. For $p=1$ we obtain the mesh independent estimate

$$||f-g||_1 \le k^{-1} \sum_{t_j\neq s_j} |t_j-s_j| \sum_{i=j-k+1}^{j} |c_i-c_{i-1}| \le 2||\boldsymbol{c}||_\infty ||\boldsymbol{t}-\boldsymbol{s}||_1.$$

We can bound a perturbation of the derivatives similarly, provided we have a bound for the derivatives of $M_{i,j}$.

LEMMA 5.2. *For any p with $1 \le p \le \infty$ the rth one sided derivative D^r of $M_{i,j}$ is bounded by*

$$||D^r M_{i,j}||_p \le \frac{\Gamma_{k,r,p}}{\delta_{i,j,k}\cdots\delta_{i,j,k-r+1}\delta_{i,j,k-r}^{1/q}}, \quad \textit{for } r=1,\ldots,k-1, \tag{5.1}$$

where

$$\Gamma_{k,r,p} = b_{r,p} \frac{(k-1)!}{(k-1-r)!\,(k-r)^{1/p}}. \tag{5.2}$$

and

$$b_{r,p} = \Big(\sum_{\nu=0}^{r} \binom{r}{\nu}^p \Big)^{1/p} \le 2^r.$$

PROOF. Arguing as in the proof of Theorem 4.22 in [4], we have

$$|D^r M_{i,j}(x)| \le \frac{(k-1)!}{(k-r-1)!} \frac{\sum_{\nu=0}^{r} \binom{r}{\nu} B_{\nu,k-r}(x)/(u_{\nu+k-r} - u_\nu)}{\delta_{i,j,k} \cdots \delta_{i,j,k-r+1}},$$

where $B_{\nu,k-r}(x) = (u_{\nu+k-r} - u_\nu)[u_\nu, \dots, u_{\nu+k-r}](\cdot - x)_+^{k-r-1}$ and $u_0, \dots, u_k$ are the knots of $M_{i,j}$, listed in nondecreasing order. Applying the L^p norm on both sides and using (2.6) this leads to

$$||D^r M_{i,j}||_p \le \frac{(k-1)!}{(k-r-1)!} \frac{\Big(\sum_{\nu=0}^{r} \binom{r}{\nu}^p (u_{\nu+k-r} - u_\nu)^{1-p}/(k-r) \Big)^{1/p}}{\delta_{i,j,k} \cdots \delta_{i,j,k-r+1}}.$$

¿From this (5.1) follows. □

Differentiating (3.2) r times, taking L^p norms, using the triangle inequality, and (5.1) we obtain the next theorem.

THEOREM 5.3. *Let r be an integer with $0 \le r \le k-1$. Then for $1 \le p \le \infty$*

$$||D^r f - D^r g||_p \le \Gamma_{k,r,p} \sum_{t_j \ne s_j} |t_j - s_j| \sum_{i=j-k+1}^{j} \frac{|c_i - c_{i-1}|}{\delta_{i,j,k} \cdots \delta_{i,j,k-r+1} \delta_{i,j,k-r}^{1/q}},$$

where $\Gamma_{k,r,p}$ is defined in Lemma 5.2 and the δ's are defined in (3.10).

Note that the bound in Theorem 5.3 reduces to that of Theorem 5.1 for $r = 0$.

The estimates in Theorem 5.1 and Theorem 5.3 are given in terms of the B-spline coefficients and the ℓ^1-norm of the difference $\boldsymbol{t} - \boldsymbol{s}$. In many cases it is more satisfactory to have a bound that is more independent of the length of the knot vector. Some of these estimates are conveniently expressed in terms of the mesh ratios

$$\rho_{i,j,k} = \frac{t_{i+k-1} - t_i}{\delta_{i,j,k}},$$
$$\rho_{i,k} = \max_{i \le j \le i+k-1} \rho_{i,j,k},$$
$$\rho_k = \max_{k \le i \le n} \rho_{i,k}.$$

If $t_i = t_{i+k-1}$ we set $\rho_{i,j,k} = 0$ irrespective of the value of $\delta_{i,j,k}$.

THEOREM 5.4. *If $f \in \mathbb{S}_{k,\boldsymbol{t}}$ and $g \in \mathbb{S}_{k,\boldsymbol{s}}$ have the same B-spline coefficients $\boldsymbol{c}$, then*

$$||f-g||_p \le \frac{2k(2L+k)}{\delta_k^{1/q}\underline{\Delta}_k^{1/p}}\kappa_{k,p}||\boldsymbol{t}-\boldsymbol{s}||_\infty ||f||_p, \tag{5.3}$$

for $1 \le p \le \infty$, where $L = L(\boldsymbol{s},\boldsymbol{t})$ is the perturbation number of $\boldsymbol{s}$ and $\boldsymbol{t}$. Alternatively,

$$||f-g||_p \le \Big(\frac{k\rho_k}{k-1}\Big)^{1/q}(2L+k-1)\kappa_{k-1,p}||\boldsymbol{t}-\boldsymbol{s}||_\infty ||Df||_p. \tag{5.4}$$

PROOF. We first prove (5.4) and start by recalling that if $f=\sum_i c_iB_{i,k,\boldsymbol{t}}$ then its derivative is $Df = \sum_i c_i^{(1)}B_{i,k-1,\boldsymbol{t}}$, where

$$c_i^{(1)} = \begin{cases} (k-1)(c_i-c_{i-1})/(t_{i+k-1}-t_i), & \text{if } t_i < t_{i+k-1}; \\ \text{arbitrary}, & \text{otherwise.} \end{cases} \tag{5.5}$$

In case Df is not continuous, it is interpreted as the one-sided derivative of f from the right.

Let now x be fixed in some interval $[t_\mu, t_{\mu+1})$ and set $I_\mu = \cup_{x\in[t_\mu,t_{\mu+1})}I(x)$, where the index set $I(x)$ is defined in Lemma 3.3. ¿From the definition of $I(x)$ we note that $I_\mu \subseteq \{\mu-L-k+2,\dots,\mu+L\}$ and therefore that $|I_\mu| \le 2L+k-1$ for all μ. From equation (3.6) in Lemma 3.3 it follows that

$$|f(x)-g(x)| \le ||\boldsymbol{t}-\boldsymbol{s}||_\infty \sum_{i\in I_\mu}|c_i^{(1)}|\sum_{j=i}^{i+k-1}M_{i,j}(x)\frac{t_{i+k-1}-t_i}{k-1}.$$

Applying p-norms on the interval $[t_\mu,t_{\mu+1})$, and making use of the second inequality in (3.12) and Hölder's inequality we find

$$\begin{aligned} ||f-g||_{L^p[t_\mu,t_{\mu+1}]} &\le ||\boldsymbol{t}-\boldsymbol{s}||_\infty \sum_{i\in I_\mu}|c_i^{(1)}|\sum_{j=i}^{i+k-1}||M_{i,j}||_p\frac{t_{i+k-1}-t_i}{k-1} \\ &\le k^{-1/p}||\boldsymbol{t}-\boldsymbol{s}||_\infty \sum_{i\in I_\mu}|c_i^{(1)}|\Big(\frac{t_{i+k-1}-t_i}{k-1}\Big)^{1/p}\sum_{j=i}^{i+k-1}\Big(\frac{\rho_{i,j,k}}{k-1}\Big)^{1/q} \\ &\le \Big(\frac{k\rho_k}{k-1}\Big)^{1/q}||\boldsymbol{t}-\boldsymbol{s}||_\infty \sum_{i\in I_\mu}|c_i^{(1)}|\Big(\frac{t_{i+k-1}-t_i}{k-1}\Big)^{1/p} \\ &\le \Big(\frac{k\rho_k}{k-1}\Big)^{1/q}||\boldsymbol{t}-\boldsymbol{s}||_\infty |I_\mu|^{1/q}\left(\sum_{i\in I_\mu}|c_i^{(1)}|^p\frac{t_{i+k-1}-t_i}{k-1}\right)^{1/p}. \end{aligned}$$

Raising this to the pth power and adding the contributions from each knot interval yields

$$||f-g||_p^p \le \left(\frac{k\rho_k}{k-1}\right)^{p/q}||\boldsymbol{t}-\boldsymbol{s}||_\infty^p(2L+k-1)^{1+p/q}||c^{(1)}||_{p,k-1,\boldsymbol{t}}^p$$

since $|I_\mu| \le 2L+k-1$ for all μ. Taking pth roots and using the stability estimate (2.6) for Df we obtain (5.4).

The first estimate (5.3) is proved in a similar way. From (3.6) we have

$$||f-g||_{L^p[t_\mu,t_{\mu+1}]} \le ||\boldsymbol{t}-\boldsymbol{s}||_\infty \sum_{i\in I_\mu} |c_i - c_{i-1}| \sum_{j=i}^{i+k-1} ||M_{i,j}||_p.$$

We set $I'_\mu = \{i_1-1\} \cup I_\mu$ where i_1 is the smallest integer in I_μ. Making use of (3.12) and Hölder's inequality as above we then find

$$\begin{aligned} ||f-g||_{L^p[t_\mu,t_{\mu+1}]} &\le 2||\boldsymbol{t}-\boldsymbol{s}||_\infty \sum_{i\in I'_\mu} |c_i| \sum_{j=i}^{i+k-1} k^{-1/p} \delta_{i,j,k}^{1/p-1} \\ &\le \frac{2k}{\delta_k^{1/q} \underline{\Delta}_k^{1/p}} ||\boldsymbol{t}-\boldsymbol{s}||_\infty \sum_{i\in I'_\mu} |c_i| \Big(\frac{t_{i+k}-t_i}{k}\Big)^{1/p} \\ &\le \frac{2k}{\delta_k^{1/q} \underline{\Delta}_k^{1/p}} ||\boldsymbol{t}-\boldsymbol{s}||_\infty |I'_\mu|^{1/q} \left(\sum_{i\in I'_\mu} |c_i|^p \Big(\frac{t_{i+k}-t_i}{k}\Big)\right)^{1/p}. \end{aligned}$$

Observing that $|I'_\mu| = 2L+k$, inequality (5.3) follows just like (5.4) □

So far we have only considered perturbations of the knots. If the B-spline coefficients are also perturbed we can combine inequality (2.9) with (5.3).

THEOREM 5.5. *Suppose $f \in \mathbb{S}_{k,\boldsymbol{t}}$ and $g \in \mathbb{S}_{k,\boldsymbol{s}}$ have B-spline coefficients $\boldsymbol{c}$ and $\boldsymbol{d}$, respectively. If $f \ne 0$ then*

$$\frac{||f-g||_p}{||f||_p} \le \kappa_{k,p} \left(2k(2L+k) \frac{||\boldsymbol{t}-\boldsymbol{s}||_\infty}{\delta_k^{1/q} \underline{\Delta}_k^{1/p}} + \frac{||\boldsymbol{c}-\boldsymbol{d}||_{p,k,t}}{||\boldsymbol{c}||_{p,k,t}} \right) \tag{5.6}$$

with δ_k given by (3.10), $\underline{\Delta}_k$ given by (2.2), $L = L(\boldsymbol{s},\boldsymbol{t})$ is the perturbation number of $\boldsymbol{s}$ and $\boldsymbol{t}$, and $||\cdot||_{p,k,\boldsymbol{t}}$ given by (2.5).

Inequality (5.6) is a generalization of (2.9) that allows perturbation of both coefficients and knots and is reminiscent of perturbation results for linear systems of equations. Since the dependence on the knots is nonlinear, it is not surprising that the part of the estimate that accounts for the influence of the knots is more involved than the part that measures the influence of the coefficients.

6 Continuity of a spline with respect to the knots

So far we have obtained a number of results that bound the difference between a spline f and the spline g obtained by perturbing the knots of f. We will now change our point of view slightly and consider the rates of convergence when the knots of g converge to the knots of f.

THEOREM 6.1. *Suppose that $f \in \mathbb{S}_{k,\boldsymbol{t}}$ and $g \in \mathbb{S}_{k,\boldsymbol{s}}$ have the same B-spline coefficients $\boldsymbol{c}$, and that $|s_i - t_i| \le \epsilon$ for $i = k+1, \ldots, n$ for some $\epsilon > 0$. Then*

$$||f-g||_p \le K_1\,\epsilon^{1/p} \tag{6.1}$$

where

$$K_1 = k^{1/q}(2L+k-1)||(c_i - c_{i-1})||_p,$$

and $L = L(\boldsymbol{s}, \boldsymbol{t})$ *is the perturbation number of* $\boldsymbol{s}$ *and* $\boldsymbol{t}$*.*

PROOF. From equation (3.6) we have

$$f(x) - g(x) = \sum_{i \in I(x)} (c_{i-1} - c_i) \sum_{j=i}^{i+k-1} (t_j - s_j) M_{i,j}(x),$$

and as before we set

$$I_\mu = \cup_{x \in [t_\mu, t_{\mu+1})} I(x).$$

Taking p-norms on the interval $[t_\mu, t_{\mu+1}]$, and using (3.12) gives

$$||f-g||_{L^p[t_\mu,t_{\mu+1}]} \le k^{-1/p} \sum_{i \in I_\mu} |c_{i-1} - c_i| \sum_{j=i}^{i+k-1} \frac{|t_j - s_j|}{\delta_{i,j,k}^{1-1/p}},$$

where $\delta_{i,j,k}$ is the support of the B-spline $M_{i,j}$ which we recall has the knots $s_i, \ldots, s_j, t_j, \ldots, t_{i+k-1}$. Arguing as in the proof of Theorem 5.4 and using the fact that $\delta_{i,j,k} \ge |t_j - s_j|$, we find

$$\begin{aligned} ||f-g||_{L^p[t_\mu,t_{\mu+1}]} &\le k^{-1/p} \sum_{i \in I_\mu} |c_{i-1} - c_i| \sum_{j=i}^{i+k-1} |t_j - s_j|^{1/p} \\ &\le k^{1-1/p} \epsilon^{1/p} \sum_{i \in I_\mu} |c_{i-1} - c_i| \\ &\le k^{1/q} \epsilon^{1/p} (2L+k-1)^{1/q} \left(\sum_{i \in I_\mu} |c_{i-1} - c_i|^p \right)^{1/p}. \end{aligned}$$

Raising this to the pth power and summing over μ on both sides we obtain the desired result. □

The following example shows that the exponent $1/p$ in $\epsilon^{1/p}$ cannot be improved in general.

EXAMPLE 6.1. Consider the two piecewise linear functions f and g defined by

$$f(x) = \begin{cases} x+1, & \text{for } -1 \le x < 0; \\ x-1, & \text{for } 0 \le x \le 1; \end{cases}$$

and

$$g(x) = \begin{cases} x+1, & \text{for } -1 \le x < 0; \\ 1 - 2x/\epsilon, & \text{for } 0 \le x < \epsilon; \\ (x-1)/(1-\epsilon), & \text{for } \epsilon \le x \le 1. \end{cases}$$

Note that f is discontinuous at $x = 0$. We can estimate $||f - g||_p$ directly by integrating. The result is

$$||f - g||_p^p = \frac{2^{p+1}\epsilon}{(p+1)(\epsilon+2)} + \frac{\epsilon^{p+2}}{(p+1)(\epsilon+2)} + \frac{\epsilon^p - \epsilon^{p+1}}{p+1}.$$

Taking the p'th root we then have

$$||f - g||_p = O(\epsilon^{1/p})$$

Thus we see that f is close to g in any L^p-norm except for $p = \infty$.

We next show that the factor $1/p$ can be removed if f and g are continuous.

THEOREM 6.2. *Suppose that $f \in \mathbb{S}_{k,\boldsymbol{t}}$ and $g \in \mathbb{S}_{k,\boldsymbol{s}}$ have the same B-spline coefficients and that both f and g are continuous. Suppose also that $|s_i - t_i| \le \epsilon$ for $i = k+1, \dots, n$ for some sufficiently small ϵ, say $\epsilon < \underline{\Delta}_{k-1}/2$. Then*

$$||f - g||_p \le K_2\,\epsilon$$

where

$$K_2 = 3k(k-1)2^k||Df||_p.$$

PROOF. By (5.4) we have

$$||f - g||_p \le \Big(\frac{k\rho_k}{k-1}\Big)^{1/q}\kappa_{k-1,p}(2L+k-1)||\boldsymbol{t}-\boldsymbol{s}||_\infty||Df||_p, \tag{6.2}$$

where L is the perturbation number. We need to bound the constants ρ_k and L. Since f is continuous we may assume that $t_{i+k-1} - t_i > 0$ for all i. By the definition of $\delta_{i,j,k}$ in (3.11) we therefore deduce that

$$\delta_{i,j,k} \ge t_{i+k-1} - s_i \ge t_{i+k-1} - t_i - \epsilon \ge \frac{t_{i+k-1} - t_i}{2}.$$

But then

$$\rho_{i,j,k} = \frac{t_{i+k-1} - t_i}{\delta_{i,j,k}} \le 2 \tag{6.3}$$

for all i and j, and hence $\rho_k \le 2$.

To bound the perturbation number we fix an arbitrary i and make use of two versions of the condition $\epsilon \le \underline{\Delta}_{k-1}/2$. Since $s_{i-k+1} - t_{i-k+1} \le (t_i - t_{i-k+1})/2$, we have

$$s_{i-k+1} \le \frac{t_{i-k+1} + t_i}{2} < t_i$$

(strict inequality because of continuity). Similarly, from the condition $t_{i+k} - s_{i+k} \le (t_{i+k} - t_{i+1})/2$ we obtain $s_{i+k} > t_{i+1}$. From this we conclude that the perturbation number satisfies $L \le k-1$. Inserting this and the bound (6.3) for ρ_k in (6.2) gives

$$||f - g||_p \le 3(k-1)^{1/p}\,(2k)^{1/q}\kappa_{k-1,p}||Df||_p\,\epsilon.$$

The result now follows from the bound in (2.10) and from the fact that the expression $2^{1+1/q}(k-1)^{1/p}k^{1/q}$ attains its maximum for $p=\infty$ and $q=1$. □

We can also show that the difference between the rth derivative of f and g is proportional to ϵ provided f and g have continuous derivatives of order $\le r$.

COROLLARY 6.3. *Suppose that $f \in \mathbb{S}_{k,\boldsymbol{t}}$ and $g \in \mathbb{S}_{k,\boldsymbol{s}}$ have the same B-spline coefficients and that $D^r f$ is continuous for $0 \le r < \ell$ while $D^\ell f$ is discontinuous. Suppose also that $|s_i - t_i| \le \epsilon$ for $i = k+1, \ldots, n$ for some $\epsilon < \underline{\Delta}_{k-\ell}/3$. Then*

$$||D^r f - D^r g||_p \le \begin{cases} K_3\,\epsilon, & \text{if } 0 \le r < \ell; \\ K_4\,\epsilon^{1/p}, & \text{if } r = \ell; \end{cases} \tag{6.4}$$

where K_3 and K_4 depend on k, n, f and $\underline{\Delta}_{k-\ell}$.

PROOF. This result follows from Theorem 5.3, but we will deduce it from the two preceding theorems. Note that it is not immediate since $D^r f$ and $D^r g$ do not in general have the same B-spline coefficients as is required by the theorems. Suppose that $\ell \ge 1$ and consider the perturbation of the first derivative. If $Df = \sum_i c^{(1)}_{i,\boldsymbol{t}} B_{i,k-1,\boldsymbol{t}}$ and $Dg = \sum_i c^{(1)}_{i,\boldsymbol{s}} B_{i,k-1\boldsymbol{s}}$ we have

$$||Df - Dg||_p \le \Big\|\sum_i (c^{(1)}_{i,\boldsymbol{t}} - c^{(1)}_{i,\boldsymbol{s}}) B_{i,k-1,\boldsymbol{t}}\Big\|_p + \Big\|\sum_i c^{(1)}_{i,\boldsymbol{s}} (B_{i,k-1,\boldsymbol{t}} - B_{i,k-1,\boldsymbol{s}})\Big\|_p. \tag{6.5}$$

For the second term on the right Theorem 6.2 or Theorem 6.1 gives the correct behaviour. Since f is assumed to be continuous we can assume that $t_i < t_{i+k-1}$ for $2 \le i \le n-1$, and since $\epsilon < \underline{\Delta}_{k-\ell}/3$ the same is true of $\boldsymbol{s}$. For the first term in (6.5) we therefore have

$$\begin{aligned} |c^{(1)}_{i,\boldsymbol{t}} - c^{(1)}_{i,\boldsymbol{s}}| &= (k-1)|c_i - c_{i-1}| \frac{|s_{i+k-1} - t_{i+k-1} + t_i - s_i|}{(t_{i+k-1} - t_i)(s_{i+k-1} - s_i)} \\ &\le 2(k-1)|c_i - c_{i-1}| \frac{\epsilon}{\underline{\Delta}_{k-1}(\underline{\Delta}_{k-1} - 2\epsilon)}. \end{aligned}$$

¿From this the result follows whether Df is continuous or not (as long as f is continuous). Arguing inductively the corollary now follows for any r with $0 \le r \le \ell$. □

Theorem 6.1 shows that even when a spline is discontinuous it is a continuous function of its knots in any L^p-space with $p < \infty$, although the continuity is only Lipschitz with exponent $1/p$. For $p = \infty$ the continuity is lost, but the difference between f and g is always bounded.

What about the continuity of the derivatives when there are knots of multiplicity k? Let us first consider the first derivative. For a B-spline for which $t_i = t_{i+k-1}$ we have by (5.5) (assuming that $s_i < s_{i+k-1}$)

$$||DB_{i,k,\boldsymbol{t}} - DB_{i,k,\boldsymbol{s}}||_p \ge (k-1)\big|\,||Q_{i,k-1,\boldsymbol{s}}||_p - ||Q_{i+1,k-1,\boldsymbol{s}} - Q_{i+1,k-1,\boldsymbol{t}}||_p\big| \tag{6.6}$$

where

$$Q_{i,k-1,\boldsymbol{t}}(x) = \begin{cases} B_{i,k-1,\boldsymbol{t}}/(t_{i+k-1} - t_i), & \text{if } t_i < t_{i+k-1}; \\ 0, & \text{otherwise.} \end{cases}$$

¿From Theorem 6.1 we know that the difference $||Q_{i+1,k-1,\boldsymbol{t}} - Q_{i+1,k-1,\boldsymbol{s}}||_p$ is bounded as $\boldsymbol{s}$ approaches $\boldsymbol{t}$ while from (3.12) we have that

$$||Q_{i,k-1,\boldsymbol{s}}||_p \geq \kappa_{k,p}^{-1}\, k^{-1/p}\, \epsilon^{1/p-1}.$$

In other words there is no convergence in the first derivative and the difference in (6.6) is unbounded in L^p for $p > 1$. For higher derivatives the difference $||D^r f - D^r g||_p$ is unbounded for all p as $\boldsymbol{s}$ approaches $\boldsymbol{t}$ when there are interior knots of multiplicity k.

7 Parametric curves

If we are working with parametric spline curves, the above analysis can be applied to each component. As an example let us consider the curve equivalent of Theorem 4.1.

COROLLARY 7.1. *If $\boldsymbol{f}$ and $\boldsymbol{g}$ are two parametric spline curves in $\mathbb{R}^d$ of order k with knots $\boldsymbol{t}$ and $\boldsymbol{s}$ and with the same B-spline coefficients $(\boldsymbol{c}_i)$ then*

$$\max_u |\boldsymbol{f}(u) - \boldsymbol{g}(u)| \leq \sum_{t_j \neq s_j} |t_j - s_j| \sum_{i=j-k+1}^{j} |\boldsymbol{c}_i - \boldsymbol{c}_{i-1}|/\delta_{i,j,k}. \tag{7.1}$$

Here $|\boldsymbol{c}_i - \boldsymbol{c}_{i-1}|$ denotes the Euclidean norm of $\boldsymbol{c}_i - \boldsymbol{c}_{i-1}$.

All the other results above can be extended in the same way. Note that there is extra freedom in the choice of vector norm. In Corollary 7.1 we used the Euclidean vector norm, but any ℓ^p norm could be used. There is also the possibility of applying a function norm to each component curve first and then applying a vector norm to the resulting vector.

Generalizing from functions to curves as we have done here has its weaknesses. Suppose for example that $k = 2$ so we are considering piecewise linear curves. If there are no multiple interior knots we know that a piecewise linear spline agrees with the piecewise linear interpolant to its B-spline coefficients. But this piecewise linear interpolant is clearly independent of the knot vector so there can be no perturbation error, whereas Corollary 7.1 would only give a positive upper bound on the perturbation error. The reason for this is of course that a parametric curve is defined as the equivalence class of all its reparametrizations. We should therefore not be comparing $\boldsymbol{f}(u)$ with $\boldsymbol{g}(u)$ in (7.1), but rather compare it with $\boldsymbol{g}\big(\phi(u)\big)$, where ϕ is an allowable change of parameter (its derivative should be nonzero). To get the best possible result we should use the change of parameter that makes the perturbation error as small as possible. In the piecewise linear case this is accomplished quite easily but letting ϕ be a suitable piecewise linear function, but in general this is a nontrivial and nonlinear problem. In any case choosing ϕ as the identity as in Corollary 7.1, does provide an upper bound for the perturbation error that in many cases should be realistic.

In certain applications the parametrization of the curve is fixed, for example when modelling the path to be followed by a camera in some system for computer animation. Then the time and speed information provided by the parametrization is significant and the perturbation result provided by Corollary 7.1 should be useful. Note that in mathematical terms this is not a parametric curve but a vector valued function.

8 Remarks

It is possible to view the perturbation of knots as a change of basis problem. Let $\boldsymbol{A_s}$ be the change of basis matrix changing from the B-splines on $\boldsymbol{s}$ to the B-splines on $\boldsymbol{s} \cup \boldsymbol{t}$ and define $\boldsymbol{A_t}$ similarly. If $f \in \mathbb{S}_{k,\boldsymbol{t}}$ and $g \in \mathbb{S}_{k,\boldsymbol{s}}$ have the same B-spline coefficients $\boldsymbol{c}$ then

$$f - g = \boldsymbol{\phi}^T(\boldsymbol{A_t} - \boldsymbol{A_s})\boldsymbol{c},$$

where $\boldsymbol{\phi}$ is the vector of B-splines of order k on $\boldsymbol{s} \cup \boldsymbol{t}$. The vectors $\boldsymbol{A_t c}$ and $\boldsymbol{A_s c}$ are discrete splines, and their difference can be bounded using the divided difference representation of discrete B-splines, see [1] and [2], combined with identities for discrete B-splines similar to the ones in Lemma 3.2 and Lemma 3.3.

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