Theory and Algorithms for Non-Uniform Spline Wavelets

T. Lyche[1)], K. Mørken[2)], and E. Quak[3)]

Abstract. We investigate mutually orthogonal spline wavelet spaces on non-uniform partitions of a bounded interval, addressing the existence, uniqueness and construction of bases of minimally supported spline wavelets. The relevant algorithms for decomposition and reconstruction are considered as well as some stability-related questions. In addition, we briefly review the bivariate case for tensor products and arbitrary triangulations. We conclude the paper with a discussion of some special cases.

§1. Introduction

Splines have become the standard mathematical tool for representing smooth shapes in computer graphics and geometric modelling. Wavelets have been introduced more recently, but are by now well established both in mathematics and in applied sciences like signal processing and numerical analysis. The two concepts are closely related as splines provide some of the most important examples of wavelets. Although there is an extensive literature on cardinal spline wavelets (spline wavelets with uniform knot spacing), see [7], relatively little has been published about spline wavelets on arbitrary, nonuniform knots, which form the subject of this paper. These kinds of wavelets, however, are needed to perform operations like decomposition, reconstruction and thresholding on splines given on a nonuniform knot vector, which typically occur in practical applications.

The flexibility of splines in modelling is due to good approximation properties, useful geometric interpretations of the B-spline coefficients, and simple algorithms for adding and removing knots. Full advantage of these capabilities can only be taken on general nonuniform knot vectors, where also multiple knots are allowed. In fact, the spline algorithms for general knots are hardly more complicated than the special ones for uniform knots. We will see in this paper that the same is true for B-wavelets which are spline wavelets of minimal support. Contrary to other types of wavelet functions, spline B-wavelets are given as explicit expressions, namely as rational functions of the knots.

The classical construction of wavelets takes place on the whole real line, with an infinite, uniform grid, but many applications require wavelets on a bounded interval. In the

[1)] Dept. of Informatics, University of Oslo, P.O.Box 1080 Blindern, 0316 Oslo, Norway, email: `tom@ifi.uio.no`.

[2)] Dept. of Informatics, University of Oslo, P.O.Box 1080 Blindern, 0316 Oslo, Norway, email: `knutm@ifi.uio.no`.

[3)] SINTEF Applied Mathematics, P.O. Box 124 Blindern, 0314 Oslo, Norway, email: `Ewald.Quak@math.sintef.no`.

early days of wavelets this problem was solved by either making the given data periodic or by extending to the whole real line, setting the data to be zero outside the interval or using reflection at the boundaries, see [13]. None of these solutions were satisfactory since they introduced various kinds of aliasing near the boundaries. Meyer proposed a better solution for the Daubechies functions in [25], namely to restrict the scaling functions and wavelets to the interval in question and then adjust the functions close to the boundary appropriately in order to preserve most of the classical properties of a multiresolution analysis. Meyer's construction suffered from certain shortcomings such as numerical instability, and was improved in [10].

The construction of general wavelets on nonuniform grids is complicated, especially in higher dimensions. The lifting scheme is a promising general framework for accomplishing this task, see [34], and also [6] for the related concept of stable completions. A brief overview of the nonuniform constructions obtained with the lifting scheme can be found in [14].

A spline multiresolution analysis on an interval with mutually orthogonal wavelet spaces can be constructed analogously to Meyer's approach, namely by retaining cardinal B-splines in the interior of the interval and by introducing special boundary splines. These boundary functions, however, are typically not obtained by restricting B-splines with uniform knots, but by introducing B-splines with multiple knots at the ends of the interval. The construction of spline wavelets in this framework was carried out in [8]. In the more general setting, where direct sum decompositions are considered instead of orthogonal ones, biorthogonal spline wavelets on the interval were investigated in [12]. In this paper, we will restrict our attention to mutually orthogonal spline wavelet spaces, but for general nonuniform knot sequences. Spline wavelets on certain types of nonuniform knots were studied in [5], and spline wavelets on general knots were first constructed in [22]. This construction was generalized to non-polynomial splines in [23]. The paper [9] discusses computations invoving spline wavelets on nonuniform, but simple, knots. For an early discussion of orthogonality in spline spaces, see [1].

The paper is organized as follows. First we summarize some necessary background material in Section 2. Then we describe in Section 3 the construction of the spline wavelets from [22]. Using the new concept of a minimal interval, we establish that these B-wavelets form the unique basis of minimal support (up to scaling) for the given wavelet space. In Section 4 we discuss algorithms for implementing the (fast) wavelet transforms and their inverses for spline wavelets on nonuniform knots. Section 5 is devoted to the bivariate setting, describing the straightforward tensor product approach and providing only a brief review of what little seems to be known for spline wavelets on arbitrary triangulations. We conclude in Section 6 with a discussion of some examples.

§2. Background Material

This section is devoted to a brief review of some basic properties of wavelets and splines that we need later. We will be working with real-valued functions defined on an interval $[a, b]$ and will use the standard L^2-norm defined by

$$||f|| = ||f||_2 = \left(\int_a^b f^2\right)^{1/2}$$

induced by the inner product

$$\langle f, g \rangle = \int_a^b fg.$$

Vectors will be denoted by bold type, and for a vector $\boldsymbol{c} = (c_1, \ldots, c_n)^T$ we will use the ℓ^2-norm

$$||\boldsymbol{c}|| = ||\boldsymbol{c}||_2 = \Big(\sum_{i=1}^n c_i^2\Big)^{1/2}.$$

We will often use vector notation for linear combinations as in

$$\sum_{i=1}^n c_i \phi_i = \boldsymbol{\phi}^T \boldsymbol{c},$$

where the functions $\{\phi_i\}_{i=1}^n$ have been gathered in the vector $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)^T$.

2.1 Multiresolution and Wavelets on an Interval

The starting point for wavelets on an interval $[a, b]$ is usually a *multiresolution analysis*. This is a nested sequence of finite dimensional subspaces of $L^2[a, b]$

$$V_0 \subset V_1 \subset \cdots$$

with the property that

$$\bigcup_{j=0}^{\infty} V_j = L^2[a, b], \tag{2.1}$$

and each space V_j is assumed to be spanned by a *Riesz basis* $\{\phi_{i,j}\}_{i=1}^{n_j}$ (the integer n_j giving the dimension of V_j), *i.e.*, the basis satisfies the inequalities

$$K_1 ||\boldsymbol{c}_j||_2 \le \Big\|\sum_i c_{i,j} \phi_{i,j}\Big\|_2 \le K_2 ||\boldsymbol{c}_j||_2,$$

for any choice of coefficients $\boldsymbol{c}_j = (c_{1,j}, \ldots, c_{n_j,j})^T$, where the two constants K_1 and K_2 are independent of j. The basis functions $\{\phi_{i,j}\}_{i=1}^{n_j}$ are often referred to as *scaling functions*.

In classical wavelet theory the space V_j is related to V_{j-1} by dilation, *i.e.*, if f is a function in V_{j-1} then $f(2\cdot)$ is a function in V_j. In our more general setting it is only assumed that each scaling function in V_{j-1} is a linear combination of the scaling functions in V_j. If we collect the scaling functions in V_j in the vector $\boldsymbol{\phi}_j = (\phi_{i,j})_i$ then this relation can be written

$$\boldsymbol{\phi}_{j-1}^T = \boldsymbol{\phi}_j^T \boldsymbol{P}_j,$$

for some matrix $\boldsymbol{P}_j$ of dimension $n_j \times n_{j-1}$. If $f = \boldsymbol{\phi}_{j-1}^T \boldsymbol{c}_{j-1}$ is a function in V_{j-1} it can then be represented in V_j as $f = \boldsymbol{\phi}_j^T \boldsymbol{c}_j$, where $\boldsymbol{c}_j = \boldsymbol{P}_j \boldsymbol{c}_{j-1}$.

The space V_j can be written as the direct sum $V_j = V_{j-1} \oplus W_{j-1}$, where W_{j-1} is the L^2-orthogonal complement of V_{j-1} in V_j. In other words W_{j-1} consists of all the functions in V_j that are orthogonal to V_{j-1},

$$W_{j-1} = \{f_j \in V_j : \langle f_j, f_{j-1} \rangle = 0 \text{ for all } f_{j-1} \in V_{j-1}\},$$

and the dimension of W_{j-1} is given by $\dim W_{j-1} = \dim V_j - \dim V_{j-1} = n_j - n_{j-1}$. By combining this decomposition repeatedly with (2.1), we obtain the direct sum decomposition

$$L^2[a,b] = V_0 \oplus W_0 \oplus W_1 \oplus \cdots. \tag{2.2}$$

We refer to the functions in the spaces $\{W_j\}_j$ as *wavelets* and the spaces as *wavelet spaces.*

For computations we need a suitable basis for each wavelet space, and we shall denote such a basis for W_{j-1} by $\boldsymbol{\psi}_{j-1} = (\psi_{i,j-1})_{i=1}^{n_j-n_{j-1}}$. Because of (2.2) we then in principle have a basis for all of $L^2[a,b]$,

$$L^2[a,b] = \operatorname{span}\{\boldsymbol{\phi}_0, \boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots\},$$

at least as long as the combined basis $(\boldsymbol{\phi}_0, \boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots)$ forms a Riesz basis, *i.e.*,

$$K_1 ||(\boldsymbol{c}_0^T, \boldsymbol{w}_0^T, \boldsymbol{w}_1^T, \ldots)||_2 \leq ||\boldsymbol{\phi}_0^T \boldsymbol{c}_0 + \sum_{j=0}^{\infty} \boldsymbol{\psi}_j^T \boldsymbol{w}_j||_2 \leq K_2 ||(\boldsymbol{c}_0^T, \boldsymbol{w}_0^T, \boldsymbol{w}_1^T, \ldots)||_2, \tag{2.3}$$

for constants K_1 and K_2.

Since $W_{j-1} \subset V_j$, the wavelet basis $\boldsymbol{\psi}_{j-1}$ is linked to $\boldsymbol{\phi}_j$ via a matrix relation $\boldsymbol{\psi}_{j-1}^T = \boldsymbol{\phi}_j^T \boldsymbol{Q}_j$. A function $g = \boldsymbol{\psi}_{j-1}^T \boldsymbol{w}_{j-1}$ can then be lifted to V_j as $g = \boldsymbol{\phi}_j^T \boldsymbol{d}_j$, where $\boldsymbol{d}_j = \boldsymbol{Q}_j \boldsymbol{w}_{j-1}$.

The relation $V_j = V_{j-1} \oplus W_{j-1}$ implies that the two bases $\boldsymbol{\phi}_{j-1}$ and $\boldsymbol{\psi}_{j-1}$ together provide an alternative basis for V_j; we have the basis transformation

$$(\boldsymbol{\phi}_{j-1}^T, \boldsymbol{\psi}_{j-1}^T) = \boldsymbol{\phi}_j^T (\boldsymbol{P}_j, \boldsymbol{Q}_j) = \boldsymbol{\phi}_j^T \boldsymbol{M}_j.$$

From the inverse relation

$$\boldsymbol{\phi}_j^T = (\boldsymbol{\phi}_{j-1}^T, \boldsymbol{\psi}_{j-1}^T) \boldsymbol{M}_j^{-1} = (\boldsymbol{\phi}_{j-1}^T, \boldsymbol{\psi}_{j-1}^T) \begin{pmatrix} \boldsymbol{A}_j \\ \boldsymbol{B}_j \end{pmatrix},$$

we obtain formulas for decomposing a function $f = \boldsymbol{\phi}_j^T \boldsymbol{c}_j$ in V_j into two components $f_{j-1} = \boldsymbol{\phi}_{j-1}^T \boldsymbol{c}_{j-1}$ and $g_{j-1} = \boldsymbol{\psi}_{j-1}^T \boldsymbol{w}_{j-1}$ in V_{j-1} and W_{j-1}, namely

$$\boldsymbol{c}_{j-1} = \boldsymbol{A}_j \boldsymbol{c}_j, \qquad \boldsymbol{w}_{j-1} = \boldsymbol{B}_j \boldsymbol{c}_j.$$

By iterating this decomposition we can rewrite a function f in V_N, say, as a coarse approximation f_0 in V_0 and a collection $\{g_j\}_{j=0}^{N-1}$ of detail functions ("wavelets") with $g_j \in W_j$ such that

$$f = f_0 + g_0 + \cdots + g_{N-1}.$$

This corresponds to a change of basis from $\boldsymbol{\phi}_N$ to $(\boldsymbol{\phi}_0, \boldsymbol{\psi}_0, \ldots, \boldsymbol{\psi}_{N-1})$ and is referred to as the (fast) *wavelet transform*. In applications the detail functions $\{g_j\}_{j=0}^{N-1}$ of f are typically filtered in some way before applying the inverse transform

$$\hat{\boldsymbol{c}}_j = \boldsymbol{P}_j \hat{\boldsymbol{c}}_{j-1} + \boldsymbol{Q}_j \hat{\boldsymbol{w}}_{j-1}, \quad \text{for } j = 1, \ldots, N,$$

where $(\hat{\boldsymbol{w}}_j)_{j=0}^{N-1}$ are the filtered wavelet coefficients and $\hat{\boldsymbol{c}}_0 = \boldsymbol{c}_0$.

In our setting in this paper, the spaces $\{V_j\}_j$ will be polynomial spline spaces with B-splines as scaling functions, see below. The matrices $\{\boldsymbol{P}_j\}_j$ are then given and the main challenge is to construct the matrices $\{\boldsymbol{Q}_j\}_j$ corresponding to suitable wavelets $(\psi_{i,j})$.

Our rudimentary introduction of wavelets has been restricted to orthogonal decompositions of the spaces $\{V_j\}_j$. More generally, one may consider biorthogonal or direct sum decompositions, but for our purposes here this is not necessary.

2.2 Splines and B-splines

We will represent splines as linear combinations of B-splines. On a given nondecreasing sequence $\boldsymbol{t} = (t_i)_{i=1}^{n+d+1}$ of knots we consider for $i = 1, \ldots, n$ the B-splines $B_{i,d} = B_{i,d,\boldsymbol{t}}$, of degree $d \geq 0$ with support $[t_i, t_{i+d+1}]$. By convention the B-splines are right continuous and are normalized to sum to one, *i.e.*,

$$\sum_{i=1}^{n} B_{i,d}(x) = 1, \quad x \in [t_{d+1}, t_{n+1}).$$

We assume that $t_i < t_{i+d+1}$ for $i = 1, \ldots, n$ so that the B-splines are linearly independent on $[t_{d+1}, t_{n+1})$. The spline space $\mathcal{S}_{d,\boldsymbol{t}}$ is the linear space spanned by these B-splines,

$$\mathcal{S}_{d,\boldsymbol{t}} = \Big\{ \sum_{i=1}^{n} c_i B_{i,d} : c_i \in \mathbb{R} \Big\}.$$

An important property of B-splines is the stability estimate (see [2])

$$D_d^{-1} ||\boldsymbol{c}||_p \leq \Big\| \sum_i c_i \tilde{B}_{i,d} \Big\|_p \leq ||\boldsymbol{c}||_p \tag{2.4}$$

for $1 \leq p \leq \infty$, where the scaled B-spline $\tilde{B}_{i,d}$ is defined by

$$\tilde{B}_{i,d} = \Big(\frac{d+1}{t_{i+d+1} - t_i} \Big)^{1/p} B_{i,d}. \tag{2.5}$$

The important point here is that the constant D_d (in addition to being independent of p) is independent of the knots $\boldsymbol{t}$. The norms $||f||_p$ and $||\boldsymbol{c}||_p$ are the standard L^p- and ℓ^p-norms for functions and vectors.

The inequalities (2.4) raise the question of what scaling to use for the B-splines in a wavelet setting. When working in L^2 the natural scaling is given by (2.5) with $p = 2$, while in most spline applications the partition of unity scaling corresponding to (2.5) with $p = \infty$ is used. We remark that this scaling can also be used in the wavelet setting provided that for highly non-uniform knot vectors we scale the coefficient matrices in the relevant linear systems so that they become well-conditioned. This will be discussed in more detail in Section 4.5.

The support of the B-spline $B_{i,d}$ is $[t_i, t_{i+d+1}]$, so the support of the spline function

$$f = \sum_{i=\ell}^{r} c_i B_{i,d}, \tag{2.6}$$

with nonzero coefficients $\boldsymbol{c}$, is the interval $[t_\ell, t_{r+d+1}]$ if $r \geq \ell$. For our purposes the notion of *index support* (often shortened to support) will prove useful. The index support of a spline gives the index of the first and last B-spline involved in its representation. If c_ℓ and c_r are nonzero, the first and last active B-splines of the spline in (2.6) are ℓ and r, respectively, and its index support is then denoted as $[\ell : r]$.

We will need some notation related to the multiplicities of knots. For each real number z, we let $m_{\boldsymbol{t}}(z)$ denote the multiplicity of z in $\boldsymbol{t}$, *i.e.*, the number of times z occurs in $\boldsymbol{t}$. The left multiplicity $\lambda_{\boldsymbol{t}}(i)$ (right multiplicity $\rho_{\boldsymbol{t}}(i)$) of a knot t_i gives the number of knots in $\boldsymbol{t}$ equal to t_i, but with index less or equal (greater or equal) than i,

$$\lambda_{\boldsymbol{t}}(i) = \max\{j : t_{i-j+1} = t_i\}, \qquad \rho_{\boldsymbol{t}}(i) = \max\{j : t_{i+j-1} = t_i\}. \tag{2.7}$$

Two sufficiently smooth functions f and g are said to agree on $\boldsymbol{t}$ if

$$D^{\lambda_{\boldsymbol{t}}(i)-1} f(t_i) = D^{\lambda_{\boldsymbol{t}}(i)-1} g(t_i) \tag{2.8}$$

for $i = 1, \ldots, n+d+1$. This corresponds to the standard interpretation of Hermite or osculatory interpolation.

Collocation matrices are a central ingredient of our constructions. If $\boldsymbol{x} = (x_i)_{i=1}^m$ are m real numbers in the interval $[a, b]$ with $a \le x_1 \le \cdots \le x_m \le b$ and $f_1, \ldots, f_n$ are n sufficiently smooth functions on $[a, b]$, we denote the $m \times n$ collocation matrix with (i, j)-entry $D^{\lambda_{\boldsymbol{x}}(i)-1} f_j(x_i)$ by

$$\begin{pmatrix} x_1, \ldots, x_m \\ f_1, \ldots, f_n \end{pmatrix}. \tag{2.9}$$

If $m = n$ we denote the determinant of the collocation matrix by

$$\det \begin{pmatrix} x_1, \ldots, x_n \\ f_1, \ldots, f_n \end{pmatrix}. \qquad (2.10).$$

The Schoenberg-Whitney theorem tells us when a square collocation matrix for splines is nonsingular, see [31].

Theorem 2.1. *With $f_j = B_{j,d,\boldsymbol{t}}(x)$ for $j = 1, \ldots, n$, the determinant of the corresponding B-spline collocation matrix (2.10) is always nonnegative and it is positive if and only if*

$$x_i \in (t_i, t_{i+d+1}) \cup \{x : D^{\lambda_{\boldsymbol{t}}(i)-1} B_{i,d,\boldsymbol{t}}(x) \neq 0\}, \quad i = 1, \ldots, n. \tag{2.11}$$

If $x_1 < x_2 < \cdots < x_n$ are distinct then condition (2.11) holds if and only if all the diagonal elements in the collocation matrix are positive.

The following lemma gives a simple criterion which ensures that a collection of spline functions are linearly independent. It is part of the folklore, and a relatively simple consequence of the local support and continuity properties of B-splines. More specifically, it is based on the fact that if $t_i = \cdots = t_{i+q} < t_{i+q+1}$ then $D^{d-q} B_{i,d}(t_i) \neq 0$ (remember that limits are taken from the right at knots).

Lemma 2.2. *Let $\{f_i\}_{i=1}^p$ be a selection of spline functions in $\mathcal{S}_{d,\boldsymbol{t}}$ with supports $\{[\ell_i : r_i]\}_{i=1}^p$ that satisfy $\ell_1 < \ell_2 < \cdots < \ell_p$ or $r_1 < r_2 < \cdots < r_p$. Then these functions are linearly independent on $[t_{\ell_1}, t_{r_p+d+1}]$.*

A basic tool for splines is knot insertion or refinement. If $\boldsymbol{\tau} = (\tau_i)_{i=1}^{n+d+1}$ and $\boldsymbol{t} = (t_i)_{i=1}^{n+m+d+1}$ are two knot vectors for splines of degree d and $\boldsymbol{\tau}$ is a subsequence of $\boldsymbol{t}$, then $\mathcal{S}_{d,\boldsymbol{\tau}} \subseteq \mathcal{S}_{d,\boldsymbol{t}}$. If we organize the two B-spline bases in the vectors $\boldsymbol{B}_{\boldsymbol{\tau}}$ and $\boldsymbol{B}_{\boldsymbol{t}}$, we therefore have the relation

$$\boldsymbol{B}_{\boldsymbol{\tau}}^T = \boldsymbol{B}_{\boldsymbol{t}}^T \boldsymbol{P} \tag{2.12}$$

for a suitable rectangular matrix $\boldsymbol{P}$ of dimension $(m+n) \times n$, the *knot insertion matrix* from $\boldsymbol{\tau}$ to $\boldsymbol{t}$. If $f = \boldsymbol{B}_{\boldsymbol{\tau}}^T \boldsymbol{c} = \boldsymbol{B}_{\boldsymbol{t}}^T \boldsymbol{b}$ is a spline lying in both spaces, it follows that $\boldsymbol{b} = \boldsymbol{P}\boldsymbol{c}$.

2.3 B-spline Gram Matrices

In the following sections, inner products of B-splines play a crucial role both in the construction of spline wavelets and in the implementation of the corresponding algorithms. Therefore we have decided to provide the reader with some more detailed information about the computation of the integrals which form the entries of B-spline Gram matrices.

First we need a notation for matrices built from inner products of functions. If $\boldsymbol{f} = (f_1, \dots, f_n)^T$ and $\boldsymbol{g} = (g_1, \dots, g_m)^T$ are two vectors of functions on $[a, b]$, we denote their Gramian by

$$\langle \boldsymbol{g}, \boldsymbol{f}^T \rangle = \begin{pmatrix} g_1, \dots, g_m \\ f_1, \dots, f_n \end{pmatrix};$$

the $m \times n$ matrix having as (i, j)-entry the number $\int_a^b g_i f_j$.

Corresponding to nondecreasing sequences of knots $(\tau_i)_{i=n_1}^{n_2}$ and $(t_i)_{i=m_1}^{m_2}$, and nonnegative integers d, e with $n_2 - n_1 \geq d + 2$ and $m_2 - m_1 \geq e + 2$ we consider the integrals

$$I_{i,j}^{d,e} = \int_{-\infty}^{\infty} B_{i,d,\boldsymbol{\tau}}(x) B_{j,e,\boldsymbol{t}}(x) dx, \quad \text{for all } i, j.$$

Since the B-splines are piecewise polynomials, one way to compute this integral is to use numerical integration on each subinterval. Thus, if $(u_1, \dots, u_{k+1})$ is an increasing arrangement of the distinct elements of

$$(\tau_i, \dots, \tau_{i+d+1}, t_j, \dots, t_{j+e+1}),$$

we have

$$I_{i,j}^{d,e} = \sum_{r=1}^{k} \int_{u_r}^{u_{r+1}} B_{i,d,\boldsymbol{\tau}}(x) B_{j,e,\boldsymbol{t}}(x) dx,$$

where each of the integrands is a polynomial of degree $d + e$ and can be computed exactly using N-point Gauss-Legendre quadrature provided $2N - 1 \geq d + e$. This method was implemented in [4] for $\boldsymbol{\tau} = \boldsymbol{t}$. It is numerically stable, but nodes and weights have to be precomputed and stored. In [27] specialized quadrature formulae using a B-spline as a weight function are considered.

An alternative to this method would be to convert the B-splines on $\boldsymbol{\tau}$ and $\boldsymbol{t}$ to their Bernstein-Bézier representations on each subinterval using knot insertion techniques. The integral on each subinterval can then be computed explicitly.

Yet another strategy is to use stable recurrence relations reminiscent of the recurrence relations for B-splines [31], and degree raising and products of B-splines [26]. Following [4], for each i, j, and positive integers k, l such that $\tau_{i+k} > \tau_i$ and $t_{j+l} > t_j$, we define quantities $T_{i,j}^{k,l}$ by

$$T_{i,j}^{k,l} = \frac{(k + l - 1)!}{(k - 1)!(l - 1)!} \int_{-\infty}^{\infty} \frac{B_{i,k-1,\boldsymbol{\tau}}(x)}{\tau_{i+k} - \tau_i} \frac{B_{j,l-1,\boldsymbol{t}}(x)}{t_{j+l} - t_j} dx. \tag{2.13}$$

We set $T_{i,j}^{k,l} = 0$ if both $\tau_{i+k} - \tau_i = 0$ and $t_{j+l} - t_j = 0$, otherwise we define

$$T_{i,j}^{k,l} = \binom{k + l - 1}{l} \frac{B_{i,k-1,\boldsymbol{\tau}}^R(t_j)}{\tau_{i+k} - \tau_i} \quad \text{if} \quad t_{j+l} = t_j,$$

$$T_{i,j}^{k,l} = \binom{k + l - 1}{l} \frac{B_{j,l-1,\boldsymbol{t}}^L(\tau_i)}{t_{j+l} - t_j} \quad \text{if} \quad \tau_{i+k} = \tau_i.$$

Here B_j^R and B_j^L are right and left continuous versions of the B-spline B_j, *i.e.*, for degree $d = 0$

$$B^R_{i,0,\boldsymbol{\tau}}(x) = \begin{cases} 1, & \text{if } \tau_j \le x < \tau_{j+1}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$B^L_{j,0,\boldsymbol{t}}(x) = \begin{cases} 1, & \text{if } t_j < x \le t_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

If $\tau_{i+k} > \tau_i$ and $t_{j+l} > t_j$ the following recurrence relations can be used for $k, l \ge 1$

$$T^{k,l}_{i,j} = \frac{(\tau_{i+k} - t_j)T^{k,l-1}_{i,j} + (t_{j+l} - \tau_{i+k})T^{k,l-1}_{i,j+1}}{t_{j+l} - t_j} + T^{k-1,l}_{i,j}, \tag{2.14}$$

$$T^{k,l}_{i,j} = \frac{(\tau_i - t_j)T^{k,l-1}_{i,j} + (t_{j+l} - \tau_i)T^{k,l-1}_{i,j+1}}{t_{j+l} - t_j} + T^{k-1,l}_{i+1,j}, \tag{2.15}$$

$$T^{k,l}_{i,j} = \frac{(t_j - \tau_i)T^{k-1,l}_{i,j} + (\tau_{i+k} - t_j)T^{k-1,l}_{i+1,j}}{\tau_{i+k} - \tau_i} + T^{k,l-1}_{i,j+1}, \tag{2.16}$$

$$T^{k,l}_{i,j} = \frac{(t_{j+l} - \tau_i)T^{k-1,l}_{i,j} + (\tau_{i+k} - t_{j+l})T^{k-1,l}_{i+1,j}}{\tau_{i+k} - \tau_i} + T^{k,l-1}_{i,j}. \tag{2.17}$$

For numerical accuracy it is advantageous to use a formula which computes the quantity on the left as a convex combination of the first two T-quantities on the right. The following list shows which formulae to use in all cases where $(\tau_i, \tau_{i+k}) \cap (t_j, t_{j+l}) \ne \emptyset$.

1. $\tau_i \le t_j \le \tau_{i+k} \le t_{j+l}$: use (2.14) or (2.16),
2. $\tau_i \le t_j \le t_{j+l} \le \tau_{i+k}$: use (2.16) or (2.17),
3. $t_j \le \tau_i \le \tau_{i+k} \le t_{j+l}$: use (2.14) or (2.15),
4. $t_j \le \tau_i \le t_{j+l} \le \tau_{i+k}$: use (2.15) or (2.17).

We conclude that in general we need all four formulae, but there are always at least two stable choices. If $\boldsymbol{\tau} = \boldsymbol{t}$ and $d = e$ one needs only two of the formulae, see [4] for a numerically stable implementation.

§3. Spline Wavelets

An important and natural example of a multiresolution analysis on an interval is provided by spline spaces. We need a collection of knot vectors $(\boldsymbol{t}^j)_{j=0}^{\infty}$ such that $\boldsymbol{t}^{j-1}$ is a subsequence of $\boldsymbol{t}^j$ for $j \ge 1$, since this ensures that $\mathcal{S}_{d,\boldsymbol{t}^{j-1}}$ is a subspace of $\mathcal{S}_{d,\boldsymbol{t}^j}$. Each space has a Riesz-basis of B-splines, and knot insertion takes us from one space to the next. Consider two nested spline spaces $\mathcal{S}_{d,\boldsymbol{\tau}}$ and $\mathcal{S}_{d,\boldsymbol{t}}$ with general knot vectors such that $\boldsymbol{\tau}$ is a subsequence of $\boldsymbol{t}$. We assume that the dimension of $\mathcal{S}_{d,\boldsymbol{\tau}}$ is n with $n \ge d+1$, and that the dimension of $\mathcal{S}_{d,\boldsymbol{t}}$ is $n+m$ so $\boldsymbol{t}$ can be thought of as obtained from $\boldsymbol{\tau}$ by inserting m *new knots* $\boldsymbol{s} = (s_i)_{i=1}^m$. For simplicity we assume that

$$\tau_1 = \tau_{d+1}, \quad \tau_{n+1} = \tau_{n+d+1}, \qquad \text{and} \qquad \tau_{d+1} < s_1 \le s_2 \le \cdots \le s_m < \tau_{n+1}, \tag{3.1}$$

the general setting is discussed in [22]. For notational convenience we rename the two sets of B-splines to

$$\begin{aligned} \phi_j &= B_{j,d,\boldsymbol{\tau}}, & \text{for } j = 1, \ldots, n, \\ \gamma_i &= B_{i,d,\boldsymbol{t}}, & \text{for } i = 1, \ldots, n+m. \end{aligned} \tag{3.2}$$

We also set

$$V_0 = \mathcal{S}_{d,\boldsymbol{\tau}} \qquad \text{and} \qquad V_1 = \mathcal{S}_{d,\boldsymbol{t}},$$

and denote the orthogonal complement of V_0 in V_1 by W.

In [22] it was shown how to construct a minimally supported basis for W. Here we give an alternative presentation of this construction that clarifies some details. From this new approach it also follows quite easily that the constructed basis is the *only* minimally supported basis for W.

The basic idea underlying the construction is simple. Any function in W must be a linear combination of B-splines in V_1. Since we want the support of a typical basis function ψ to be minimal, we want a linear combination of as few consecutive B-splines in V_1 as possible, say q altogether. The constraint is that ψ must be orthogonal to all the s coarse B-splines whose supports intersect the support of ψ. Intuitively, the support of ψ should be minimal when the number of parameters is one greater than the number of conditions, *i.e.*, when $q = s + 1$, and this turns out to be the case. An extra bonus is that in this case it is easy to write down an explicit formula for ψ. In this way the construction of minimally supported wavelets is reduced to the problem of finding minimal intervals that can support a wavelet, and we show that the total number of such intervals agrees exactly with the dimension of W.

3.1 Construction of Minimally Supported Wavelets

A function ψ in V_1 with index support $[\ell : r]$, where $r \geq \ell$, is a linear combination of the $r - \ell + 1$ fine B-splines $\boldsymbol{\gamma} = (\gamma_\ell, \ldots, \gamma_r)^T$,

$$\psi = \sum_{j=\ell}^{r} w_j \gamma_j = \boldsymbol{\gamma}^T \boldsymbol{w},$$

and is zero off the interval $[t_\ell, t_{r+d+1}]$. Let us assume that there are $p = p(\ell, r) \geq 0$ old knots in the open interval (t_ℓ, t_{r+d+1}), namely $\tau_{k+1}, \ldots, \tau_{k+p}$, for some $k = k(\ell, r)$. There are then $p + d + 1$ coarse B-splines $\boldsymbol{\phi} = (\phi_{k-d}, \ldots, \phi_{k+p})^T$, whose supports intersect the interval (t_ℓ, t_{r+d+1}). Due to the local supports of B-splines, the spline ψ is orthogonal to V_0 if and only if

$$\langle \phi_i, \psi \rangle = 0, \quad \text{for } i = k - d, \ldots, k + p,$$

or in terms of the Gram matrix

$$\langle \boldsymbol{\phi}, \boldsymbol{\gamma}^T \rangle \boldsymbol{w} = 0.$$

As the orthogonality conditions are homogeneous, a non-trivial solution is guaranteed to exist if the number of parameters is strictly larger than the number of conditions imposed, namely $r - \ell + 1 > p(\ell, r) + d + 1$. In the hope of obtaining a wavelet ψ with smallest possible support we want this inequality to be satisfied with $r - \ell$ as small as possible. This motivates the following definition.

Definition 3.1. *The index interval* $[\ell : r]$ *is said to be* minimal *if*

$$r - \ell > p(\ell, r) + d \tag{3.3}$$

and there is no true subinterval $[u : v]$ *with this property, i.e., if* $\ell \le u < v \le r$ *and* $v - u > p(u, v) + d$*, then* $u = \ell$ *and* $v = r$.

A minimal interval always exists for given $\boldsymbol{\tau}$ and $\boldsymbol{t}$, as the condition (3.3) is certainly satisfied for the index interval $[1 : n + m]$; a smallest subinterval for which (3.3) is true must therefore be minimal. Also since a minimal interval cannot contain other minimal intervals than itself we observe that if $[\ell_1 : r_1]$ and $[\ell_2 : r_2]$ are two distinct minimal intervals then

$$(\ell_2 - \ell_1)(r_2 - r_1) > 0. \tag{3.4}$$

Lemma 3.2. *For a minimal interval* $[\ell : r]$ *we have*

$$r - \ell = p(\ell, r) + d + 1, \tag{3.5}$$

so that for minimal intervals the number of parameters is exactly one larger than the number of orthogonality conditions imposed.

Proof: Suppose the contrary is true, *i.e.*, that $[\ell : r]$ is minimal, but $r - \ell > p(\ell, r) + d + 1$. We distinguish between two cases. First, if $t_{\ell+1} < \tau_{k+1}$, we have $p(\ell + 1, r) = p(\ell, r)$. But then $r - (\ell + 1) > p(\ell + 1, r) + d$, contradicting the minimality of $[\ell : r]$.

If, however, $t_{\ell+1} = \tau_{k+1}$ and τ_{k+1} is a knot of multiplicity $s \ge 1$ in the knot sequence $\boldsymbol{\tau}$, then we have $p(\ell + 1, r) = p(\ell, r) - s$, and therefore $r - (\ell + 1) > p(\ell + 1, r) + d + s$, again contradicting the minimality of $[\ell : r]$. $\square$

Our aim is to construct a wavelet with as small a support as possible, so let us clarify what exactly we mean by this.

Definition 3.3. *A nonzero function* ψ *in* V_1 *with support* $[u : v]$ *is called a B-wavelet if it is orthogonal to* $\mathcal{S}_{d,\boldsymbol{\tau}}$*, and if it has minimal support in the sense that if* g *is another nonzero spline in* W *with support* $[\tilde{u} : \tilde{v}]$*, where* $u \le \tilde{u} \le \tilde{v} \le v$*, then* $u = \tilde{u}$ *and* $v = \tilde{v}$.

Our next result shows that any minimal interval $[\ell : r]$ is the support interval of a B-wavelet.

Theorem 3.4. *Let* $[\ell : r]$ *be a minimal interval. Then the spline function* $\psi(x) = \sum_{i=\ell}^{r} w_i \gamma_i(x)$ *given by*

$$\psi(x) = \det \begin{pmatrix} \langle \phi_{k-d}, \gamma_\ell \rangle & \langle \phi_{k-d}, \gamma_{\ell+1} \rangle & \cdots & \langle \phi_{k-d}, \gamma_r \rangle \\ \vdots & \vdots & & \vdots \\ \langle \phi_{k+p}, \gamma_\ell \rangle & \langle \phi_{k+p}, \gamma_{\ell+1} \rangle & \cdots & \langle \phi_{k+p}, \gamma_r \rangle \\ \gamma_\ell(x) & \gamma_{\ell+1}(x) & \cdots & \gamma_r(x) \end{pmatrix} \tag{3.6}$$

is a B-wavelet with index support $[\ell : r]$*, its coefficients* $(w_i)_{i=\ell}^{r}$ *oscillate strictly in sign, and* ψ *itself has* $r - \ell$ *strong changes of sign in* (t_ℓ, t_{r+d+1}).

Proof: Expanding the determinant (3.6), we find that $\psi(x) = \sum_{i=\ell}^{r} w_i \gamma_i(x)$, where $w_i = (-1)^{r-i} v_i$, $v_i = \det G_i$, and

$$G_i = \begin{pmatrix} \phi_{k-d}, \ldots, \phi_{k+p} \\ \gamma_\ell, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_r \end{pmatrix}. \tag{3.7}$$

Consequently, the spline ψ lies in V_1 and its support is included in $[\ell : r]$. Taking inner products of ψ in (3.6) with any of the coarse B-splines $\{\phi_{k+j}\}_{j=-d}^{p}$ yields a determinant with two identical rows and therefore establishes that ψ is indeed orthogonal to V_0.

We next show that $v_\ell > 0$. By a formula from [18, p.17] we have

$$v_\ell = \int_\Omega \det \begin{pmatrix} y_1, & \dots, & y_{r-\ell} \\ \phi_{k-d}, & \dots, & \phi_{k+p} \end{pmatrix} \det \begin{pmatrix} y_1, & \dots, & y_{r-\ell} \\ \gamma_{\ell+1}, & \dots, & \gamma_r \end{pmatrix}, \tag{3.8}$$

where Ω denotes the set

$$\Omega = \{(y_1, \dots, y_{r-\ell}) : a \le y_1 \le \dots \le y_{r-\ell} \le b\}.$$

Theorem 2.1 immediately implies that $v_\ell \ge 0$. Since the B-splines are continuous functions, it is clear that v_ℓ will be positive if the integrand is positive at some point in the interior of Ω. But again, from Theorem 2.1, we know that this happens if

$$\text{supp}_0 \phi_{k-d-1+i} \cap \text{supp}_0 \gamma_{\ell+i} \ne \emptyset, \quad \text{for } i = 1, \dots, r-\ell, \tag{3.9}$$

where $\text{supp}_0 f$ denotes the interior of the support of f. For $i = r - \ell$, we have that $\text{supp}_0 \phi_{k-d-1+i} = (\tau_{k+p}, \tau_{k+p+d+1})$ and $\text{supp}_0 \gamma_r = (t_r, t_{r+d+1})$. Since $\tau_{k+p} \in (t_\ell, t_{r+d+1})$ but $\tau_{k+p+1} \ge t_{r+d+1}$, we see that the condition in (3.9) is satisfied in this case.

Suppose now that (3.9) does not hold for some integer $i = j$ with $1 \le j < r - \ell$, in other words

$$(\tau_{k-d-1+j}, \tau_{k+j}) \cap (t_{\ell+j}, t_{\ell+j+d+1}) = \emptyset.$$

As condition (3.9) holds for $i = r - \ell$, this is only possible if $\tau_{k+j} \le t_{\ell+j}$. But since $\boldsymbol{\tau} \subset \boldsymbol{t}$, this implies that

$$\tau_{k+i} \le t_{\ell+i}, \quad \text{for } i = j, j-1, \dots, 1.$$

In particular, for $i = 1$ we then have

$$t_\ell < \tau_{k+1} = t_{\ell+1}.$$

From Lemma 3.2, we have $r - \ell = p(\ell, r) + d + 1$, but if τ_{k+1} has multiplicity $s \ge 1$ in $\boldsymbol{\tau}$, then $p(\ell+1, r) = p(\ell, r) - s$, and

$$r - (\ell + 1) = p(\ell+1, r) + d + s > p(\ell+1, r) + d,$$

contradicting the minimality of $[\ell : r]$. The conclusion is therefore that $v_\ell > 0$, and by an analogous proof we conclude that $v_i > 0$ for $i = \ell+1, \dots, r$.

It remains to show that ψ is a B-wavelet. Note that $v_\ell > 0$ implies that the matrix G_ℓ is non-singular, so the only linear combination of $\gamma_{\ell+1}, \dots, \gamma_r$ that satisfies all the orthogonality conditions is the trivial one. Similarly, since the last coefficient v_r is also positive, the sub-matrix G_r is also nonsingular, which shows that there cannot be a non-trivial combination of $\gamma_\ell, \dots, \gamma_{r-1}$ in W. This shows that ψ has minimal support.

That ψ has $r - \ell$ strong sign changes in (t_ℓ, t_{r+d+1}) follows from Lemma 2 in [22]. □

Note at this point that as a consequence of the integral formulas of Section 2.3, the B-wavelet coefficients are in fact rational functions of the knots.

We cannot have more minimal intervals than new knots.

Lemma 3.5. *The B-wavelets in W generated by different minimal intervals are linearly independent and there are at most m of them.*

Proof: Suppose that we have s B-wavelets with index supports $[\ell_j : r_j]$ for $j = 1, \ldots, s$. Equation (3.4) implies that these B-wavelets may be ordered so that $\ell_1 < \ell_2 < \cdots < \ell_s$, and from Lemma 2.2 we therefore know that they are linearly independent. But since $\dim W = m$ it follows that $s \leq m$. □

As we shall see shortly, there are exactly m minimal intervals, so the corresponding B-wavelets form a basis for W. The clue to this fact is an explicit procedure for constructing minimal intervals, one for each new knot.

The equation $r - \ell - p(\ell, r) = d + 1$ implied by Lemma 3.2 for a minimal interval can be rewritten as

$$r + d + 1 - \ell + 1 = p(\ell, r) + 2d + 3.$$

This relation can be interpreted as stating that, apart from the $p(\ell, r)$ old knots in (t_ℓ, t_{r+d+1}), there are $2d + 3$ knots in the index interval $[\ell : r]$. More precisely, there must be $2d + 3 - \rho_{\boldsymbol{t}}(\ell) - \lambda_{\boldsymbol{t}}(r)$ new knots s_i in (t_ℓ, t_{r+d+1}). As the multiplicity of any given knot is at most $d + 1$, there has to be at least one such new knot.

If s_j is a new knot and $t_{\tilde{\ell}}$ and $t_{\tilde{r}}$ are two knots with $t_{\tilde{\ell}} < s_j < t_{\tilde{r}}$ we define the integer functions μ_j and ν_j by

$$\mu_j(\tilde{\ell}) = \#\{i < j : s_i \in (t_{\tilde{\ell}}, s_j]\},$$
$$\nu_j(\tilde{r}) = \#\{i > j : s_i \in [s_j, t_{\tilde{r}})\}.$$

We see that if $[\ell : r]$ is a minimal interval and s_j is a new knot in (t_ℓ, t_{r+d+1}) then the relation

$$\rho_{\boldsymbol{t}}(\ell) + \mu_j(\ell) + \nu_j(r + d + 1) + \lambda_{\boldsymbol{t}}(r + d + 1) = 2d + 2 \tag{3.10}$$

must hold (the sum is $2d + 2$ since s_j is not counted in (3.10)). However, equation (3.10) is not sufficient to ensure minimality, but the following proposition gives a 'symmetric' construction of a minimal interval.

Proposition 3.6. *Let s_j be a new knot, let ℓ be the largest integer less than j such that*

$$\mu_j(\ell) + \rho_{\boldsymbol{t}}(\ell) = d + 1, \tag{3.11}$$

and let r be the smallest integer greater than j such that

$$\nu_j(r + d + 1) + \lambda_{\boldsymbol{t}}(r + d + 1) = d + 1. \tag{3.12}$$

Then $[\ell : r]$ is a minimal interval.

Proof: We first show that the construction is well defined in that integers ℓ and r that satisfy (3.11) and (3.12) exist. That (3.11) can be satisfied follows from three simple facts:

(i) $\mu_j(1) + \rho_{\boldsymbol{t}}(1) \geq d + 1$.

(ii) If t_q is the largest q such that $t_q < s_j$, then $\mu_j(q) + \rho_{\boldsymbol{t}}(q) \leq d + 1$.

(iii) If $t_p < s_j$ and $p > 1$ then $\mu_j(p-1) + \rho_{\boldsymbol{t}}(p-1) \le \mu_j(p) + \rho_{\boldsymbol{t}}(p) + 1$, so the left-hand side in (3.11) increases by at most one when ℓ is reduced by one.

The first fact is immediate since $\rho_{\boldsymbol{t}}(1) = d+1$, while the second follows since no knots occur more than $d+1$ times in $\boldsymbol{t}$. The third fact is obvious if $t_{p-1} = t_p$, while if $t_{p-1} < t_p$ it follows from the three relations $\mu_j(p-1) \le m_{\boldsymbol{t}}(t_p) + \mu_j(p)$, $\rho_{\boldsymbol{t}}(p-1) = 1$ and $\rho_{\boldsymbol{t}}(p) = m_{\boldsymbol{t}}(t_p)$. The existence of an integer r that satisfies (3.12) follows similarly.

The selection of ℓ and r ensures that $r - \ell = p(\ell, r) + 2d + 2$, but not yet that $r - \ell$ is minimal. So suppose that $[\ell : r]$ contains a minimal subinterval $[\ell' : r']$. Without loss we assume that $\ell < \ell'$ and $r' \le r$. If $s_j \in (t_{\ell'}, t_{r'+d+1})$ we must have

$$\mu_j(\ell') + \rho_{\boldsymbol{t}}(\ell') < d + 1$$

since ℓ was chosen as the largest integer satisfying (3.11). Likewise we must have

$$\nu_j(r' + d + 1) + \lambda_{\boldsymbol{t}}(r' + d + 1) \le d + 1,$$

so

$$\rho_{\boldsymbol{t}}(\ell') + \mu_j(\ell') + \nu_j(r' + d + 1) + \lambda_{\boldsymbol{t}}(r' + d + 1) < 2d + 2$$

contradicting (3.10) and thereby the minimality of $[\ell' : r']$.

Another possibility is that $s_j \le t_{\ell'}$. As we noted above, there must be at least one new knot, say s_i, in $(t_{\ell'}, t_{r'+d+1})$. Since s_i also lies in (t_ℓ, t_{r+d+1}) we must have

$$\rho_{\boldsymbol{t}}(\ell) + \mu_i(\ell) = d + 1 - j + i,$$
$$\nu_i(r + d + 1) + \lambda_{\boldsymbol{t}}(r + d + 1) = d + 1 + j - i,$$

and ℓ and r must be the largest and smallest integers that satisfy these equations respectively. But if $\ell < \ell'$ and $r' \le r$ we must have

$$\rho_{\boldsymbol{t}}(\ell') + \mu_i(\ell') < d + 1 - j + i$$
$$\nu_i(r' + d + 1) + \lambda_{\boldsymbol{t}}(r' + d + 1) \le d + 1 + j - i,$$

so

$$\rho_{\boldsymbol{t}}(\ell') + \mu_i(\ell') + \nu_i(r' + d + 1) + \lambda_{\boldsymbol{t}}(r' + d + 1) < 2d + 2$$

which again contradicts the minimality of $[\ell' : r']$. The final possibility is that $s_j \ge t_{r'+d+1}$ and this can be treated similarly. We therefore conclude that $[\ell : r]$ is minimal. □

By applying the construction in Proposition 3.6 to all m new knots we obtain m minimal intervals, and each of these gives rise to a B-wavelet as stated in Theorem 3.4.

Theorem 3.7. *Let $[\ell_j : r_j]$ be the m minimal intervals obtained by applying the construction in Proposition 3.6 to each of the m new knots $(s_j)_{j=1}^m$, with $s_1 \le \cdots \le s_m$, and let ψ_j denote the B-wavelet associated with $[\ell_j : r_j]$ for $j = 1, \ldots, m$. Then it holds that $\ell_1 < \cdots < \ell_m$, $r_1 < \cdots < r_m$, the m associated B-wavelets $\{\psi_j\}_{j=1}^m$ are the only B-wavelets in W (apart from scaling), and these B-wavelets form a basis for this space.*

Proof: For fixed j with $2 \le j \le m$ we first show that $\ell_{j-1} < \ell_j$ and $r_{j-1} < r_j$. Since $s_{j-1} \le s_j$ it follows from the definition of the ℓ's and r's that $\ell_{j-1} \le \ell_j$ and $r_{j-1} \le r_j$. Suppose that $\ell_{j-1} = \ell_j = \ell$. By construction we then have

$$\mu_{j-1}(\ell) + \rho_{\boldsymbol{t}}(\ell) = d + 1 = \mu_j(\ell) + \rho_{\boldsymbol{t}}(\ell).$$

But then $\mu_{j-1}(\ell) = \mu_j(\ell)$ which is impossible by the definition of the μ's. Similarly it follows that $r_{j-1} < r_j$.

It remains to be shown that there are no other B-wavelets in W. Suppose that ψ is a B-wavelet. Since the B-wavelets $\{\psi_j\}_{j=1}^m$ form a basis for W we have

$$\psi = \sum_{j=j_1}^{j_2} c_j \psi_j$$

for certain numbers $(c_j)_{j=j_1}^{j_2}$ with c_{j_1} and c_{j_2} nonzero. But since the left and right ends of the index supports of $\{\psi_j\}_{j=1}^m$ are strictly increasing, we see that the index support of ψ is $[\ell_{j_1} : r_{j_2}]$. But since ψ is a minimally supported wavelet in W this forces $j_1 = j_2$ and $\psi = c\psi_{j_1}$ for some nonzero constant c. □

Lemma 11 in [22] gives the following alternative representation of a B-wavelet.

Lemma 3.8. *Let $[\ell : r]$ be a minimal interval, let ψ be the associated B-wavelet, and let $\tau_{k+1}, \ldots, \tau_{k+p}$ be the old knots in (t_ℓ, t_{r+d+1}). Then $\psi = D^{d+1}\theta$, where θ is the spline of degree $2d+1$ given by*

$$\theta(x) = e \det \begin{pmatrix} x, \tau_{k+1}, \ldots, \tau_{k+p} \\ B_{\ell,2d+1,\boldsymbol{t}}, \ldots, B_{\ell+p,2d+1,\boldsymbol{t}} \end{pmatrix},$$

and e is a nonzero constant.

This lemma shows that B-wavelets can, if desirable, be computed from smaller determinants than the one used in (3.6). Also since $\ell + p = r - d - 1$ the $p+1$ B-splines $(B_{i,2d+1,\boldsymbol{t}})_{i=\ell}^{\ell+p}$ of degree $2d+1$ only depend on the knots $t_\ell, \ldots t_{r+d+1}$.

Finally, we note that polynomial reproduction of degree d is obvious in our spline setting since the spline spaces contain polynomials of degree d and the approximation method from the fine space to the coarse space is orthogonal projection. In other words, polynomials of degree d can be projected from the fine space to the coarse space with zero error, i.e., the detail component in W will be zero. But this in turn means that all of W is orthogonal to polynomials of degree d, and in particular we have $\langle \psi, x^i \rangle = 0$ for $i = 0, \ldots, d$.

§4. Algorithms for Decomposition and Reconstruction

To implement the complete wavelet transform, we must first decide how to choose the nested sequence of spaces $\{V_j\}_j$. If we are given a spline f in a fine space V_N, this means that we must determine which of the interior knots from V_N to keep in V_{N-1}, then which knots from V_{N-1} to keep in V_{N-2} and so on. We consider some different approaches to selecting subspaces and then discuss reconstruction and decomposition using the B-wavelets that we constructed in Section 3, see also [9].

4.1 Choosing the Subspaces

The simplest choice is to keep every other knot in analogy with classical wavelets. An obvious generalization is to choose a fraction μ/ν and keep μ out of every ν knots. The advantage of this approach is its simplicity.

An alternative is to use information about f when deciding which knots to pass on to V_{N-1}, as in the knot removal strategy in [21]. In this strategy, given a spline function and a tolerance, the aim is to remove as many knots from the spline as possible, without the error exceeding the tolerance. The knots are first ranked according to their significance in the representation of the spline. As many knots as possible are then removed, in the order of increasing significance taking some uniformity constraints into account. The knots of the resulting approximation are then ranked again and removed according to the same criteria. This process is continued until no more knots can be removed.

Knot removal is clearly closely related to compression via wavelet decomposition, in that we compute in both cases successive approximations in coarser and coarser spaces. The difference is that in knot removal, the detail (error) functions are not stored, although they are computed (and represented in the fine space) to check the magnitude of the error. Using knot removal it is possible to ensure that the approximation in V_{N-1} will have certain desirable properties, for example that the approximation is so good that the error (the wavelet component of f) may be neglected. On the other hand, it may be argued that this mixes two stages of the wavelet analysis: the decomposition and the analysis. For efficiency, the choice of the spaces in the wavelet decomposition ought to be independent of f, but the following analysis must of course involve (the decomposed version of) f.

In the ranking part of knot removal in [21] each knot is given a weight indicating its significance. Since each new knot is associated with a spline wavelet one could instead use the wavelet coefficient as a weight for the corresponding knot. In particular, it is possible to use a wavelet decomposition in a knot removal strategy. We simply carry out a wavelet transform on some nested sequence of spaces, defined by, say, removing every second knot. The wavelets which remain after the thresholding using a certain tolerance then define the knots which are candidates for knots to keep.

4.2. Basis Transformations

Our aim here is to sketch efficient algorithms for working with the spline wavelets that we constructed in Section 3. We use the same notation and let V_0 and V_1 be coarse and fine spline spaces of dimension n and $n+m$, spanned by B-splines of degree d that we organize in two vectors $\boldsymbol{\phi}$ and $\boldsymbol{\gamma}$. The complement space W of dimension m is spanned by the m B-wavelets $\boldsymbol{\psi} = (\psi_j)_{j=1}^m$.

Together the two bases $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ form a basis for V_1 so we have the standard wavelet relation

$$(\boldsymbol{\phi}^T, \boldsymbol{\psi}^T) = \boldsymbol{\gamma}^T(\boldsymbol{P}\ \boldsymbol{Q}) = \boldsymbol{\gamma}^T \boldsymbol{M} \tag{4.1}$$

and its inverse

$$\boldsymbol{\gamma}^T = (\boldsymbol{\phi}^T, \boldsymbol{\psi}^T)\boldsymbol{M}^{-1} = (\boldsymbol{\phi}^T, \boldsymbol{\psi}^T)\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix}. \tag{4.2}$$

The matrix $\boldsymbol{P}$ is the knot insertion matrix, while the matrix $\boldsymbol{Q}$ shows how the B-wavelets are related to the fine B-splines. It can be obtained from the explicit representation for

B-wavelets in equation (3.6). When $\boldsymbol{P}$ and $\boldsymbol{Q}$ are known, the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are also given, as a decomposition of the inverse of $\boldsymbol{M}$.

Due to the relation (4.1), any function $f_1 \in V_1$ has two different representations with uniquely determined coefficient vectors, namely

$$f_1 = \boldsymbol{\gamma}^T \boldsymbol{c}_1 = f_0 + g = \boldsymbol{\phi}^T \boldsymbol{c}_0 + \boldsymbol{\psi}^T \boldsymbol{w}.$$

The computation of the coarse spline coefficients $\boldsymbol{c}_0$ and the wavelet coefficients $\boldsymbol{w}$, when given the fine spline coefficients $\boldsymbol{c}_1$, is referred to as *decomposition*, while the converse procedure is referred to as *reconstruction*.

Reconstruction is accomplished via the reconstruction relation

$$\boldsymbol{c}_1 = \boldsymbol{M} \begin{pmatrix} \boldsymbol{c}_0 \\ \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P} & \boldsymbol{Q} \end{pmatrix} \begin{pmatrix} \boldsymbol{c}_0 \\ \boldsymbol{w} \end{pmatrix} = \boldsymbol{P}\boldsymbol{c}_0 + \boldsymbol{Q}\boldsymbol{w}, \tag{4.3}$$

whereas decomposition is based on the inverse relation

$$\begin{pmatrix} \boldsymbol{c}_0 \\ \boldsymbol{w} \end{pmatrix} = \boldsymbol{M}^{-1}\boldsymbol{c}_1 = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{pmatrix} \boldsymbol{c}_1 = \begin{pmatrix} \boldsymbol{A}\,\boldsymbol{c}_1 \\ \boldsymbol{B}\,\boldsymbol{c}_1 \end{pmatrix}. \tag{4.4}$$

Let us first see how we can perform reconstruction.

4.3 Reconstruction

Once the two matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are known, reconstruction is straightforward with the relation (4.3), and since both $\boldsymbol{P}$ and $\boldsymbol{Q}$ in many cases are banded matrices, this is also efficient.

The matrix $\boldsymbol{P}$ can be computed efficiently by the Oslo algorithm, see [11,20]. Our primary concern here is therefore the efficient computation of $\boldsymbol{Q}$. From the relation $\boldsymbol{\psi}^T = \boldsymbol{\gamma}^T \boldsymbol{Q}$ we see that column j of $\boldsymbol{Q}$ gives the B-spline coefficients of ψ_j relative to the fine B-splines. More specifically, the jth B-wavelet ψ_j with support $[\ell_j : r_j]$ is given as in (3.6) with $\ell = \ell_j$, $r = r_j$ and $p = p_j = p(\ell_j, r_j)$. Note however that we are free to choose a different scaling of ψ_j. Due to the alternating coefficients of ψ_j according to Theorem 3.4, we have the following

Lemma 4.1. *Let $[\ell_j : r_j]$ be the index support of the jth B-wavelet ψ_j^*, scaled such that the absolute values of its B-spline coefficients add to one. Let $p_j = p(\ell_j, r_j)$ be the number of old knots in $(t_{\ell_j}, t_{r_j+d+1})$. The nonzero B-spline coefficients $(q_{i,j})_{i=\ell_j}^{r_j}$ of ψ_j^* are given by the solution of the linear system*

$$\begin{pmatrix} \langle \phi_{k-d}, \gamma_{\ell_j} \rangle & \langle \phi_{k-d}, \gamma_{\ell_j+1} \rangle & \cdots & \langle \phi_{k-d}, \gamma_{r_j} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \phi_{k+p_j}, \gamma_{\ell_j} \rangle & \langle \phi_{k+p_j}, \gamma_{\ell_j+1} \rangle & \cdots & \langle \phi_{k+p_j}, \gamma_{r_j} \rangle \\ 1 & -1 & \cdots & (-1)^{r_j-\ell_j} \end{pmatrix} \begin{pmatrix} q_{\ell_j,j} \\ \vdots \\ q_{r_j-1,j} \\ q_{r_j,j} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{4.5}$$

and constitute the nonzero part of column j of $\boldsymbol{Q}$.

The linear system (4.5) can be solved by straightforward Gaussian elimination without pivoting since the matrix obtained by deleting the last row is totally positive. Recall that the computation of the inner product entries can be carried out as discussed in Section 2.3. Determination of $\boldsymbol{Q}$ is therefore straightforward once a scaling has been chosen. We will discuss scaling in some more detail in Section 4.5.

4.4 Decomposition

The algorithm for reconstruction is straightforward, but it is not so simple to devise a good decomposition algorithm. A direct computation of the inverse $\boldsymbol{M}^{-1}$, and thereby of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ needed in (4.4), results in full matrices with corresponding loss of efficiency. The entries of these matrices typically decay away from the main diagonals, and so one might suggest setting all matrix elements below a certain threshold to zero in order to obtain banded matrices. However, practical experience shows that such a truncation leads to significant round-off errors.

An alternative is to treat the equation

$$(\boldsymbol{P} \quad \boldsymbol{Q}) \begin{pmatrix} \boldsymbol{c}_0 \\ \boldsymbol{w} \end{pmatrix} = \boldsymbol{c}_1 \tag{4.6}$$

as a linear system with $(\boldsymbol{P} \quad \boldsymbol{Q})$ as the coefficient matrix of size $n+m$, with the given coefficients $\boldsymbol{c}_1$ in V_1 as the right hand side, and with $\boldsymbol{c}_0$ and $\boldsymbol{w}$ as the unknowns. If Gaussian elimination is used to solve this linear system we need to use some kind of reordering of equations and/or unknowns. This is because the matrix $(\boldsymbol{P} \quad \boldsymbol{Q})$ will typically have zero diagonal elements. One possibility is to reorder the unknowns and interlace the columns of $\boldsymbol{P}$ and $\boldsymbol{Q}$ to obtain a new (globally) banded coefficient matrix, allowing the use of a special banded system solver. Exactly how to do this interlacing is not clear, but it seems reasonable to place column i of $\boldsymbol{Q}$ in a position such that its nonzero entries are distributed as evenly as possible above and below the diagonal.

A third approach is based on the fact that due to the orthogonality of the decomposition $f_1 = f_0 + g$, the functions f_0 and g are in fact the least squares best approximations to f_1 from V_0 and W, respectively, and can therefore be computed using normal equations. The normal equations can be derived quite simply. The wavelet function (the error) $g = f_1 - f_0$ should be orthogonal to V_0, *i.e.*, to each coarse B-spline. Since $f_1 = \boldsymbol{\gamma}^T \boldsymbol{c}_1$ and $f_0 = \boldsymbol{\phi}^T \boldsymbol{c}_0$ these conditions can be expressed as a system of n linear equations in n unknowns,

$$\langle \boldsymbol{\phi}, \boldsymbol{\phi}^T \rangle \boldsymbol{c}_0 = \langle \boldsymbol{\phi}, \boldsymbol{\gamma}^T \rangle \boldsymbol{c}_1. \tag{4.7}$$

The other set of normal equations are derived from the conditions that the approximation $f_0 = f_1 - g$ should be orthogonal to W, or equivalently, all the B-wavelets in W. With $g = \boldsymbol{\psi}^T \boldsymbol{w}$ this leads to m linear equations in the m unknowns $\boldsymbol{w}$,

$$\langle \boldsymbol{\psi}, \boldsymbol{\psi}^T \rangle \boldsymbol{w} = \langle \boldsymbol{\psi}, \boldsymbol{\gamma}^T \rangle \boldsymbol{c}_1. \tag{4.8}$$

To solve the systems (4.7) and (4.8) the four matrices involved must be computed. The two matrices $\langle \boldsymbol{\phi}, \boldsymbol{\phi}^T \rangle$ and $\langle \boldsymbol{\phi}, \boldsymbol{\gamma}^T \rangle$, consisting of inner-products of B-splines, can be computed directly, using one of the techniques described in Section 2.3. The other two matrices are most conveniently computed by making use of the refinement equation (4.1) and relating the matrices to the matrix $\langle \boldsymbol{\gamma}, \boldsymbol{\gamma}^T \rangle$ of inner-products of fine B-splines,

$$\langle \boldsymbol{\psi}, \boldsymbol{\psi}^T \rangle = \boldsymbol{Q}^T \langle \boldsymbol{\gamma}, \boldsymbol{\gamma}^T \rangle \boldsymbol{Q}, \qquad \langle \boldsymbol{\psi}, \boldsymbol{\gamma}^T \rangle = \boldsymbol{Q}^T \langle \boldsymbol{\gamma}, \boldsymbol{\gamma}^T \rangle.$$

For computational efficiency it is important to exploit the bandedness of all the matrices involved.

4.5 Stability Considerations

It is common to require from a wavelet construction that the combined set of basis functions $(\boldsymbol{\phi}_0, \boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots)$ form a Riesz basis as in (2.3), since this ensures that we have a basis for all of $L^2[a,b]$ and that computations with the combined basis can be performed without excessive loss of accuracy. For spline wavelets there are several interpretations of 'Riesz basis'. If the nested spaces $\{V_j\}_{j=0}^{\infty}$ are spline spaces of degree d, generated by knot vectors $(\boldsymbol{t}^j)_{j=0}^{\infty}$, the required inequalities are

$$K_1||(\boldsymbol{c}_0^T, \boldsymbol{w}_0^T, \boldsymbol{w}_1^T, \ldots)||_2 \leq ||\boldsymbol{\phi}_0^T \boldsymbol{c}_0 + \sum_{j=0}^{\infty} \boldsymbol{\psi}_j^T \boldsymbol{w}_j||_2 \leq K_2||(\boldsymbol{c}_0^T, \boldsymbol{w}_0^T, \boldsymbol{w}_1^T, \ldots)||_2, \tag{4.9}$$

where K_1 and K_2 are two constants such that $K_1^{-1}K_2$ is not overly large. However, in the case of splines there is one added complication in that the constants may depend on the knots. It is therefore possible that we have a Riesz basis in the case where each new interior knot is inserted half way between two old knots, but that there are certain knot configurations where one or both of the constants become infinite. The ultimate form of stability would of course be that the constants are completely independent of the knots, as is the case for B-splines, see (2.4). To our knowledge, very little is known about the stability of spline wavelets at present; in fact it is not even known whether we have stability in the simplest nonuniform case where the knot intervals are halved each time.

In spite of the lack of results on stability of the wavelet construction, the stability of the B-spline basis also has consequences for wavelet computations. Recall that for B-splines $\boldsymbol{B}_{\boldsymbol{\tau}} = (B_{i,d})_{i=1}^n$ on a knot vector $\boldsymbol{\tau}$ we have the stability estimate

$$D_d^{-1}||\boldsymbol{c}||_2 \leq \left\|\tilde{\boldsymbol{B}}_{\boldsymbol{\tau}}^T \boldsymbol{c}\right\|_2 \leq ||\boldsymbol{c}||_2, \tag{4.10}$$

where the scaled B-spline $\tilde{B}_{i,d}$ is defined by

$$\tilde{B}_{i,d} = \Big(\frac{d+1}{\tau_{i+d+1} - \tau_i}\Big)^{1/2} B_{i,d}. \tag{4.11}$$

Since

$$||\tilde{\boldsymbol{B}}_d^T \boldsymbol{c}||_2^2 = \boldsymbol{c}^T \langle \tilde{\boldsymbol{B}}_d, \tilde{\boldsymbol{B}}_d^T \rangle \boldsymbol{c},$$

this implies that the largest eigenvalue of the Gram matrix $\langle \tilde{\boldsymbol{B}}_d, \tilde{\boldsymbol{B}}_d^T \rangle$ is 1, while the smallest is D_d^{-2}, so the condition number of the Gram matrix is D_d^2. In other words, if the B-splines are scaled as in (4.11), then the corresponding Gram matrix is always well conditioned, while if the standard scaling is used, the condition number may be proportional to the ratio

$$\frac{\max_i(\tau_{i+d+1} - \tau_i)}{\min_i(\tau_{i+d+1} - \tau_i)}.$$

This means that if the wavelet decompositions are computed by solving normal equations, the coefficient matrix of the system (4.7) is well conditioned if the B-splines are scaled as in (4.11).

When working with B-splines it is convenient, however, to work with the partition of unity normalization and rather scale the coefficient matrices in the linear systems so that they become well-conditioned. We introduce an $n \times n$ diagonal matrix $\boldsymbol{E}_{\boldsymbol{\tau}}$ with diagonal elements $(\tau_{i+d+1} - \tau_i)/(d+1)$ for $i = 1, \ldots, n$ and a similar $(n+m) \times (n+m)$ diagonal matrix $\boldsymbol{E}_{\boldsymbol{t}}$. Instead of (4.7) we can then solve the system

$$\boldsymbol{E}_{\boldsymbol{\tau}}^{-1/2} \langle \boldsymbol{\phi}, \boldsymbol{\phi}^T \rangle \boldsymbol{E}_{\boldsymbol{\tau}}^{-1/2} \big(\boldsymbol{E}_{\boldsymbol{\tau}}^{1/2} \boldsymbol{c}_0 \big) = \boldsymbol{E}_{\boldsymbol{\tau}}^{-1/2} \langle \boldsymbol{\phi}, \gamma^T \rangle \boldsymbol{c}_1,$$

which by the above analysis has a condition number bounded independently of the knot vectors.

The knot insertion matrix $\boldsymbol{P}$ can also be scaled such that its condition number is bounded. As before, we let $\boldsymbol{t} = (t_i)_{i=1}^{n+m}$ be a refined knot vector which contains $\boldsymbol{\tau}$ as a subsequence, and we let $\boldsymbol{P}$ be the matrix that relates the two bases as in (2.12). In [21] it was shown that two more inequalities can be fitted into (4.10),

$$D_d^{-1} ||\boldsymbol{c}||_2 \le ||\tilde{\boldsymbol{B}}_{\boldsymbol{\tau}}^T \boldsymbol{c}||_2 \le ||\tilde{\boldsymbol{P}} \boldsymbol{c}||_2 \le ||\boldsymbol{c}||_2. \tag{4.12}$$

Here $\tilde{\boldsymbol{P}}$ is the scaled version of $\boldsymbol{P}$ given by

$$\tilde{\boldsymbol{P}} = \boldsymbol{E}_{\boldsymbol{t}}^{1/2} \boldsymbol{P} \boldsymbol{E}_{\boldsymbol{\tau}}^{-1/2}.$$

From the inequalities in (4.12) it follows that the condition number of $\tilde{\boldsymbol{P}}$ (defined as the ratio between its largest and smallest singular values) is bounded by D_d. This in turn means that some care should be observed when computing with the matrix $\langle \boldsymbol{\phi}, \boldsymbol{\gamma}^T \rangle = \boldsymbol{P} \langle \gamma, \boldsymbol{\gamma}^T \rangle$. The lack of stability results for the matrix $\boldsymbol{Q}$ means that we have no guarantee that the system (4.8) can be solved accurately.

Our first suggestion for decomposition was to solve (4.6). Even if we use the scaled version $\tilde{\boldsymbol{P}}$ of $\boldsymbol{P}$, we cannot guarantee the stability of the system as long as the stability of $\boldsymbol{Q}$ is not under control.

So far we have only discussed stability with respect to the L^2-norm, but since the B-spline basis, scaled correctly, is stable in any L^p-space, see (2.4), one may try to establish stability in other norms as well. Stability with respect to the L^∞-norm would be of particular interest. Gram matrices are naturally related to the L^2-norm, so it is more difficult to prove that the systems (4.7) and (4.8) are stable in L^∞. For the system (4.6) it is known that $\boldsymbol{P}$ (without any further scaling) has bounded L^∞ condition number, and it appears as likely that properly scaled versions of $\boldsymbol{Q}$ should be stable both in L^∞ as in L^2.

§5. Tensor Products and Triangulations

In this section we consider briefly the bivariate setting, namely tensor product spline wavelets and (piecewise linear) spline wavelets on arbitrary bounded triangulations.

5.1. Tensor Product Spline Wavelets

The generalization to the tensor-product setting is fairly straightforward, but is included for the sake of completeness. Let $V_1 = V_0 \oplus W$ be one decomposition of a spline space as previously described, with corresponding matrices $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{M}$, etc. and

$$\bar{V}_1 = \bar{V}_0 \oplus \bar{W}$$

another one with dimensions $\bar{n}, \bar{m}$, and $\bar{n}+\bar{m}$, bases $\bar{\boldsymbol{\gamma}}$, $\bar{\boldsymbol{\phi}}$, and $\bar{\boldsymbol{\psi}}$, and matrices $\bar{\boldsymbol{P}}, \bar{\boldsymbol{Q}}, \bar{\boldsymbol{M}}$, etc.

Then we have the decomposition of the tensor-product space as

$$V_1 \times \bar{V}_1 = \big(V_0 \times \bar{V}_0\big) \oplus \big(V_0 \times \bar{W}\big) \oplus \big(W \times \bar{V}_0\big) \oplus \big(W \times \bar{W}\big),$$

i.e., three mutually orthogonal wavelet spaces representing functions that are a) a coarse spline in the first and a wavelet in the second component, b) a wavelet in the first and a coarse spline in the second component, and c) a wavelet in both components.

A function f_1 in $V_1 \times \bar{V}_1$ can now be written using an $(n+m) \times (\bar{n}+\bar{m})$ coefficient matrix $\boldsymbol{C}^1$, namely

$$f_1(x,y) = \boldsymbol{\gamma}(x)^T \boldsymbol{C}_1 \bar{\boldsymbol{\gamma}}(y).$$

Similarly,

$$\begin{aligned} f_0(x,y) &= \boldsymbol{\phi}(x)^T \boldsymbol{C}^0 \bar{\boldsymbol{\phi}}(y) \in V_0 \times \bar{V}_0 \\ g^{(1)}(x,y) &= \boldsymbol{\phi}(x)^T \boldsymbol{D}^{(1)} \bar{\boldsymbol{\psi}}(y) \in V_0 \times \bar{W} \\ g^{(2)}(x,y) &= \boldsymbol{\psi}(x)^T \boldsymbol{D}^{(2)} \bar{\boldsymbol{\phi}}(y) \in W \times \bar{V}_0 \\ g^{(3)}(x,y) &= \boldsymbol{\psi}(x)^T \boldsymbol{D}^{(3)} \bar{\boldsymbol{\psi}}(y) \in W \times \bar{W}. \end{aligned}$$

The matrix multiplication formula for reconstruction then becomes

$$\boldsymbol{C}^1 = \boldsymbol{P}(\boldsymbol{C}^0 \bar{\boldsymbol{P}}^T + \boldsymbol{D}^{(1)} \bar{\boldsymbol{Q}}^T) + \boldsymbol{Q}(\boldsymbol{D}^{(2)} \bar{\boldsymbol{P}}^T + \boldsymbol{D}^{(3)} \bar{\boldsymbol{Q}}^T).$$

Direct decomposition means first solving $\bar{n}+\bar{m}$ linear systems of size $n+m$ with $\boldsymbol{M} = (\boldsymbol{P}\ \boldsymbol{Q})$ as coefficient matrix and the right hand sides given by the columns of $\boldsymbol{C}^1$, as in

$$(\,\boldsymbol{P} \quad \boldsymbol{Q}\,)\,\boldsymbol{R} = \boldsymbol{C}^1.$$

Splitting the resulting $(n+m) \times (\bar{n}+\bar{m})$ matrix $\boldsymbol{R}$ into two parts of sizes $n \times (\bar{n}+\bar{m})$ and $m \times (\bar{n}+\bar{m})$, *i.e.*,

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_s \\ \boldsymbol{R}_w \end{pmatrix}$$

shows that the second phase consists of first solving n systems of size $\bar{n}+\bar{m}$ with coefficient matrix $\bar{\boldsymbol{M}} = (\bar{\boldsymbol{P}}\ \bar{\boldsymbol{Q}})$,

$$(\,\bar{\boldsymbol{P}} \quad \bar{\boldsymbol{Q}}\,) \begin{pmatrix} (\boldsymbol{C}^0)^T \\ (\boldsymbol{D}^{(1)})^T \end{pmatrix} = \boldsymbol{R}_s^T$$

in order to determine $\boldsymbol{C}^0$ and $\boldsymbol{D}^{(1)}$, and then solving m systems of size $\bar{n}+\bar{m}$, again with coefficient matrix $\bar{\boldsymbol{M}}$,

$$(\,\bar{\boldsymbol{P}} \quad \bar{\boldsymbol{Q}}\,)\begin{pmatrix} \left(\boldsymbol{D}^{(2)}\right)^T \\ \left(\boldsymbol{D}^{(3)}\right)^T \end{pmatrix} = \boldsymbol{R}_w^T$$

to compute $\boldsymbol{D}^{(2)}$ and $\boldsymbol{D}^{(3)}$.

On the other hand, the normal equation approach amounts to solving the matrix equations

$$\begin{aligned} \boldsymbol{F}\boldsymbol{C}^0\bar{\boldsymbol{F}} &= \boldsymbol{P}^T\boldsymbol{G}\boldsymbol{C}^1\bar{\boldsymbol{G}}\bar{\boldsymbol{P}} \\ \boldsymbol{F}\boldsymbol{D}^{(1)}\bar{\boldsymbol{H}} &= \boldsymbol{P}^T\boldsymbol{G}\boldsymbol{C}^1\bar{\boldsymbol{G}}\bar{\boldsymbol{Q}} \\ \boldsymbol{H}\boldsymbol{D}^{(2)}\bar{\boldsymbol{F}} &= \boldsymbol{Q}^T\boldsymbol{G}\boldsymbol{C}^1\bar{\boldsymbol{G}}\bar{\boldsymbol{P}} \\ \boldsymbol{H}\boldsymbol{D}^{(3)}\bar{\boldsymbol{H}} &= \boldsymbol{Q}^T\boldsymbol{G}\boldsymbol{C}^1\bar{\boldsymbol{G}}\bar{\boldsymbol{Q}} \end{aligned}$$

for $\boldsymbol{C}^0$, $\boldsymbol{D}^{(1)}$, $\boldsymbol{D}^{(2)}$, and $\boldsymbol{D}^{(3)}$. Here $\boldsymbol{F} = \langle\boldsymbol{\phi},\boldsymbol{\phi}^T\rangle$, $\bar{\boldsymbol{F}} = \langle\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\phi}}^T\rangle$, $\boldsymbol{H} = \langle\boldsymbol{\psi},\boldsymbol{\psi}^T\rangle$, and $\bar{\boldsymbol{H}} = \langle\bar{\boldsymbol{\psi}},\bar{\boldsymbol{\psi}}^T\rangle$ are univariate Gram matrices. As in the direct decomposition, this is achieved by solving linear systems, in this case in the first stage with the matrices $\boldsymbol{F}$ and $\boldsymbol{H}$ as coefficient matrices, and with the corresponding matrices $\bar{\boldsymbol{F}}$ and $\bar{\boldsymbol{H}}$ in the second.

5.2. Spline Wavelets on Triangulations

The bivariate non-tensor-product setting, *i.e.*, the construction of compactly supported piecewise polynomial spline wavelets on triangulations, poses considerable challenges. This is especially true if the given triangulation is supposed to be bounded and arbitrary, and not an infinite uniform one, for which wavelets based on box splines have been constructed, [3]. Based on subdivision schemes it is possible to construct so-called surface wavelets, see [19], even for surface triangulations of arbitrary topology in 3D. These surface wavelets, however, possess in general no closed-form polynomial spline representations.

In the remainder of this section, we will consider only planar bounded triangulations in the following sense. A *triangle* is the convex hull of three non-collinear points $[x_1, x_2, x_3]$ in $\mathbb{R}^2$, and this triangle has three edges $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$. Let $\mathcal{T}_0 = \{T_1, \ldots, T_M\}$ be a set of triangles and let $\Omega = \bigcup_{i=1}^{M} T_i$ be their union. Then we say that $\mathcal{T}_0$ is a `triangulation` if

(i) the intersection $T_i \cap T_j$ is either empty or corresponds to a common vertex or a common edge, $i \neq j$,
(ii) the number of boundary edges incident on a boundary vertex is two,
(iii) the region Ω is simply connected.

Associated with $\mathcal{T}_0$ we associate the set $\mathcal{E}_0$ which consists of all edges of triangles in $\mathcal{T}_0$, and the set $\mathcal{V}_0$ which consists of all the vertices of $\mathcal{T}_0$. By a *boundary vertex* or *boundary edge* we mean a vertex or edge contained in the boundary of Ω. All other vertices and edges are *interior*. For a vertex $v \in \mathcal{V}_0$, we denote its *set of neighbours* in $\mathcal{V}_0$ by

$$\mathcal{N}_v = \{w \in \mathcal{V}_0 : [v, w] \in \mathcal{E}_0\}.$$

While it is possible to describe very general refinement procedures in the univariate setting we described earlier, it is necessary to restrict our attention to uniform refinement of a triangulation in order to obtain the explicit results outlined later on.

Given a triangulation $\mathcal{T}_0$ we consider its *uniform refinement* $\mathcal{T}_1$, the triangulation formed by dividing each triangle $[x_1, x_2, x_3]$ in $\mathcal{T}_0$ into four congruent subtriangles. Specifically, if y_1, y_2, y_3 are the midpoints of the edges $[x_2, x_3]$, $[x_3, x_1]$, $[x_1, x_2]$ respectively, then the subtriangles are

$$[x_1, y_2, y_3], \quad [y_1, x_2, y_3], \quad [y_1, y_2, x_3], \quad [y_1, y_2, y_3].$$

We clearly have $\mathcal{V}_0 \subset \mathcal{V}_1$, and if we set $\mathcal{V}_* = \mathcal{V}_1 \setminus \mathcal{V}_0$, we see that a vertex in $\mathcal{V}_1$ is either a *coarse vertex*, namely an element of $\mathcal{V}_0$, or a *fine vertex*, an element of $\mathcal{V}_*$.

Contrary to the univariate setting, where the polynomial degree of the splines was arbitrary, we are only aware of explicit results on triangular spline wavelets of polynomial degree 1, *i.e.*, piecewise linear functions.

Let V_j be the linear space of piecewise linear functions over $\mathcal{T}_j$, in other words, the set of functions which are linear over each triangle in $\mathcal{T}_j$ and continuous over Ω. It is then clear that

$$V_0 \subset V_1.$$

As basis functions for the spaces V_0 and V_1, respectively, we choose the *nodal* (or *hat*) functions, namely for each $v \in \mathcal{V}_0$, $\phi_v \in V_0$ such that $\phi_v(w) = \delta_{vw}$ for $w \in \mathcal{V}_0$, and analogously, for each $v \in \mathcal{V}_1$, $\gamma_v \in V_1$ such that $\gamma_v(w) = \delta_{vw}$ for $w \in \mathcal{V}_1$. The basis sets of nodal functions are then $\boldsymbol{\phi} = \{\phi_v\}_{v \in \mathcal{V}_0}$ and $\boldsymbol{\gamma} = \{\gamma_v\}_{v \in \mathcal{V}_1}$. Their cardinalities, and thus the dimensions of the spaces V_0 and V_1, are $|\mathcal{V}_0|$ and $|\mathcal{V}_1|$, respectively.

Note that the support of ϕ_v is the union of all triangles which contain the vertex $v \in \mathcal{V}_0$, called the *cell of* v, which we denote by

$$\mathcal{C}_v := \bigcup_{\substack{T \in \mathcal{T}_0 \\ v \in T}} T.$$

The refinement relation relating the two bases is given by

$$\phi_v = \gamma_v + \frac{1}{2} \sum_{w \in \mathcal{N}_v} \gamma_w, \qquad v \in \mathcal{V}_0.$$

We can define an inner product $\langle \cdot, \cdot \rangle$ on $L^2(\Omega)$ by

$$\langle f, g \rangle = \sum_{T \in \mathcal{T}_0} \frac{1}{\mathrm{a}(T)} \int_T f(x) g(x)\, dx, \qquad f, g \in L^2(\Omega), \tag{5.1}$$

where $\mathrm{a}(T)$ is the area of triangle T. With respect to this (weighted) inner product, the spaces V_j become Hilbert spaces, with corresponding (weighted) 2-norm $||f||_2 = \langle f, f \rangle^{1/2}$. This norm is equivalent to the unweighted L^2-norm. In principle, it is also possible to use the standard, nonweighted inner product, and obtain similar results, but the use of the inner product in (5.1) results in a considerable reduction of the computations necessary to determine the wavelet coefficients, which is especially important for applications in computer graphics (see [19]).

Let W denote the relative orthogonal complement of the coarse space V_0 in the fine space V_1 with respect to the inner product (5.1), so that

$$V_1 = V_0 \oplus W.$$

The dimension of W is $|\mathcal{E}_0| = |\mathcal{V}_1| - |\mathcal{V}_0|$. Therefore, a basis of W is typically described by associating one basis element with each edge in the triangulation $\mathcal{T}_0$, or equivalently, with each midpoint of such an edge.

If we let the coarse vertices $v, v^* \in \mathcal{V}_0$ be the endpoints of the edge $[v, v^*] \in \mathcal{E}_0$, which has the midpoint $u \in \mathcal{V}_*$, we note that $u \in \mathcal{N}_v \cap \mathcal{N}_{v^*}$. Finding a wavelet basis $\{\psi_u\}_{u\in\mathcal{V}_*}$ of W thus amounts to first describing the construction of an element $\psi_u \in V_1$ that actually lies in W, and then showing that the whole set of these elements in W is linearly independent. Additionally, the total collection should be stable, *i.e.*, satisfy inequality (2.3).

The surface wavelet construction in [19], when specialized to this planar piecewise linear setting, starts with a fine hat function γ_u. Letting $L_0(\gamma_u)$ denote the orthogonal projection of γ_u onto V_0 using the inner product (5.1), the difference $\gamma_u - L_0(\gamma_u)$ is a wavelet in W. This function, however, has global support on all of Ω, but for computational reasons it would be desirable to have a locally supported basis for W.

We are aware of only two (different and independently developed) approaches for obtaining a basis for W with local support. The one in [32] originates from the numerical treatment of PDEs and can be generalized to higher dimensions, but produces piecewise linear spline wavelets, whose supports are somewhat larger than for the wavelets studied in [16] and [17]. The latter construction is based on an approach to compute directly the orthogonality conditions that are imposed for minimally supported elements of the wavelet space.

More specifically, the two different strategies share the common approach that for each coarse vertex $v \in \mathcal{V}_0$ some kind of auxiliary function in V_1 is defined whose support lies in the cell $\mathcal{C}_v$ and which satisfies most, but not all orthogonality conditions needed for an element in W. A linear combination of auxiliary functions corresponding to several coarse vertices is then used to produce an element in V_1 satisfying all orthogonality conditions, *i.e.*, a true wavelet.

The approach in [32] uses an auxiliary function $\tilde{\phi}_v \in V_1$ for a given vertex $v \in \mathcal{V}_0$, with support in $\mathcal{C}_v$, that is orthogonal to all but one coarse hat function, namely the one for v itself, *i.e.*,

$$\tilde{\phi}_v \perp \phi_w, \quad w \in \mathcal{V}_0, \quad w \neq v.$$

In addition to the endpoints of the edge $[v, v^*]$ containing u, let $a, b \in \mathcal{V}_0$ be the other two coarse vertices of the quadrilateral in $\mathcal{T}_0$ containing $[v, v^*]$, if the edge is interior, and just let $a \in \mathcal{V}_0$ be the remaining vertex of the triangle in $\mathcal{T}_0$ containing $[v, v^*]$, in case the edge is a boundary one. A wavelet ψ_u^* is then constructed in [32] by a linear combination of the fine hat function γ_u and the auxiliary functions $\tilde{\phi}_v, \tilde{\phi}_{v^*}, \tilde{\phi}_a$, and $\tilde{\phi}_b$ in the interior case, and just $\tilde{\phi}_v, \tilde{\phi}_{v^*}$, and $\tilde{\phi}_a$ in the boundary case. Consequently,

$$\operatorname{supp}(\psi_u^*) \subset (\mathcal{C}_v \cup \mathcal{C}_{v^*} \cup \mathcal{C}_a \cup \mathcal{C}_b)\,,$$

or in the boundary case,

$$\operatorname{supp}(\psi_u^*) \subset (\mathcal{C}_v \cup \mathcal{C}_{v^*} \cup \mathcal{C}_a)\,.$$

Different auxiliary functions $\sigma_{v,u}$ for a coarse vertex $v \in \mathcal{V}_0$ and a fine vertex $u \in [v, v^*]$ are called semi-wavelets in [16]. Here, the support of $\sigma_{v,u} \in V_1$ is also included in $\mathcal{C}_v$, but the function satisfies all but two orthogonality conditions, namely the ones imposed by v and v^*,

$$\sigma_{v,u} \perp \phi_w, \quad w \in \mathcal{V}_0, \quad w \neq v, v^*.$$

Then it is possible to just add $\sigma_{v,u}$ and $\sigma_{v^*,u}$ to obtain a true wavelet ψ_u, resulting in a smaller support than for Stevenson's approach, as

$$\operatorname{supp}(\psi_u) \subset (\mathcal{C}_v \cup \mathcal{C}_{v^*}).$$

Note that the dimension of the space of all wavelets whose support lies in $\mathcal{C}_v \cup \mathcal{C}_\sqcap$ is larger than one. In fact, the dimension of the space of all wavelets whose support is contained in $\mathcal{C}_v \cup \mathcal{C}_{v^*}$, is not only larger than one, but the dimension even depends on the specific local topology of the triangulation, see [15]. This maybe leaves a larger choice of basis elements, but it also becomes possible to make a wrong choice, *i.e.*, one can easily find a choice of a nonzero element in W for each $u \in [v, v^*]$, with support in $\mathcal{C}_v \cup \mathcal{C}_{v^*}$, so that the total collection does not form a basis. In [16] it is shown that it is in fact possible to choose specific elements ψ_u of the given support, so that the total collection $\{\psi_u\}_{u \in \mathcal{V}_*}$ is a stable basis. The algorithmic aspects of this particular approach, including examples from terrain modelling, are presented in [17].

§6. Discussion of Some Examples

Perhaps the simplest situation is the one where $\boldsymbol{t}$ is obtained from $\boldsymbol{\tau}$ by adding one knot. The corresponding wavelet space is onedimensional and the only B-wavelet is globally supported. The wavelet coefficient of a given function f then gives an indication of the importance of the new knot for the representation of f.

In recent years there has been considerable interest in so-called multiwavelets. Spline multiwavelets result from wavelet constructions on uniform knot vectors where each knot has a fixed multiplicity r. If the knots are at the integers, the complete B-spline basis is generated by r distinct B-splines and their integer translates. This is clearly a special case of our construction. The classical wavelet theory generalizes nicely to multiwavelets, see for example [33] and the references therein.

In some cases, for example for closed parametric curves, it is useful to deal with splines which are periodic on some interval $[a, b]$. For $n, d \in \mathbb{N}$ with $n \geq d + 2$ let $\boldsymbol{t} = (t_j)_{j=1}^{n+2d+1}$ be a knot sequence with $t_{d+1} = a$, $t_{n+d+1} = b$ and $t_{i+n} = t_i + b - a$ for $i = 1, 2, \ldots, 2d + 1$. On this $(b - a)$-periodic knot sequence we can define periodic B-splines which form a basis for a space of periodic splines. The dimension of this space is n. We refer to [31] for further details. Spline wavelets can be constructed for nested periodic spline spaces, see [24] for the quadratic, trigonometric spline case. In general the matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ will be as in the non-periodic case except that we get a few corner elements due to wraparound of some columns.

The standard adaptation of any kind of wavelet approach to an interval, as mentioned already in the introduction, is to keep most of the interior scaling and wavelet functions, *i.e.*, functions whose supports lie completely within the interval. Special boundary scaling and wavelet functions are then introduced using approaches depending on the specific type of multiresolution analysis. For spline wavelets of degree d, this means that uniform dyadic knots are used in the interior of the interval, with knots of multiplicity $d+1$ at the interval endpoints, *i.e.*, for nonnegative integers d, n the knot sequence is

$$\boldsymbol{u}_{d,n} = \{\overbrace{0,\ldots,0}^{d+1}\} \cup \{\frac{k}{2^n} : 1 \le k \le 2^n - 1\} \cup \{\overbrace{1,\ldots,1}^{d+1}\}.$$

This setting was first considered in [8]. The corresponding decomposition and reconstruction algorithms were investigated in [28], their tensor-product versions in [29], and a spline wavelet packet approach in [30].

As a typical and very frequently used example, let us consider the cubic case $d = 3$, for which we want to provide all numerical values needed to implement the algorithms of Section 4. Consider specifically the knot sequences $\boldsymbol{\tau} = \boldsymbol{u}_{3,3}$ and its uniform refinement $\boldsymbol{t} = \boldsymbol{u}_{3,4}$. The corresponding spline spaces V_0 and V_1 are of course nested, have dimensions 11 and 19, respectively, and contain all polynomials of degree 3. The corresponding matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ thus have dimensions 19×11 and 19×8 and are given below. These matrices contain the information that is necessary also for any $n \ge 4$, as further refinement only leads to more interior columns, while the number of boundary columns as well as their entries remain unchanged.

$$16\boldsymbol{P} = \begin{pmatrix}
16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 12 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 11 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 12 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 12 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 12 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 12 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 11 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 12 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16
\end{pmatrix},$$

$$
\boldsymbol{Q} = \begin{pmatrix}
-\frac{1136914560}{27877} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1655323200}{27877} & \frac{9450650880}{1381667} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1321223960}{27877} & -\frac{77583612430}{4145001} & -\frac{153545}{396} & 0 & 0 & 0 & 0 & 0 \\
\frac{633094403}{27877} & \frac{409599117799}{16580004} & \frac{6643465}{3168} & 1 & 0 & 0 & 0 & 0 \\
-\frac{229000092}{27877} & -\frac{36869700393}{1381667} & -\frac{738445}{88} & -124 & 0 & 0 & 0 & 0 \\
\frac{46819570}{27877} & \frac{157389496903}{8290002} & \frac{29839177}{1584} & 1677 & 1 & 0 & 0 & 0 \\
-124 & -\frac{32916268667}{4145001} & -\frac{19335989}{792} & -7904 & -124 & 0 & 0 & 0 \\
1 & \frac{27809640281}{16580004} & \frac{58651607}{3168} & 18482 & 1677 & 1 & 0 & 0 \\
0 & -124 & -\frac{521819}{66} & -24264 & -7904 & -124 & 0 & 0 \\
0 & 1 & \frac{442733}{264} & 18482 & 18482 & \frac{442733}{264} & 1 & 0 \\
0 & 0 & -124 & -7904 & -24264 & -\frac{521819}{66} & -124 & 0 \\
0 & 0 & 1 & 1677 & 18482 & \frac{58651607}{3168} & \frac{27809640281}{16580004} & 1 \\
0 & 0 & 0 & -124 & -7904 & -\frac{19335989}{792} & -\frac{32916268667}{4145001} & -124 \\
0 & 0 & 0 & 1 & 1677 & \frac{29839177}{1584} & \frac{157389496903}{8290002} & \frac{46819570}{27877} \\
0 & 0 & 0 & 0 & -124 & -\frac{738445}{88} & -\frac{36869700393}{1381667} & -\frac{229000092}{27877} \\
0 & 0 & 0 & 0 & 1 & \frac{6643465}{3168} & \frac{409599117799}{16580004} & \frac{633094403}{27877} \\
0 & 0 & 0 & 0 & 0 & -\frac{153545}{396} & -\frac{77583612430}{4145001} & -\frac{1321223960}{27877} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{9450650880}{1381667} & \frac{1655323200}{27877} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1136914560}{27877}
\end{pmatrix}
$$

Note that the entries of $\boldsymbol{Q}$ near the boundary differ somewhat from those derived in [28]. There, the boundary spline wavelets were constructed in order to preserve as many coefficients from the interior elements as possible. Consequently, the boundary spline wavelets in [28] are not B-wavelets, in the sense that their index supports are not minimal. Their actual support intervals, however, are the same as for the B-wavelets.

The matrix $\boldsymbol{Q}$ as given here offers also a nice illustration of the as yet unsolved problem of how the various B-wavelets, *i.e.*, the columns of $\boldsymbol{Q}$, should be scaled. Here, more or less arbitrarily and just to save some space, each column has 1 and -124 as its first (or last) two nonzero entries. To achieve the scaling as in Lemma 4.1, for example, the two interior columns need to be multiplied by $1/80640$, the first and last one by $27877/5025860410$, the second and next to last by $82900002/876051996025$, and the third and third to last by $1584/130442935$.

In this special case, it is possible to determine a strategy how to interlace the columns of P and Q to obtain a banded square matrix, allowing to implement decomposition by solving the (permuted) system by means of a special banded system solver. For each column of P and Q, respectively, the index of the row which contains the element of largest absolute value is determined. This index then provides the column number for the overall square matrix where the original column is to be placed. For the given example matrices, the columns of P thus become columns $1, 3, 4, 6, 8, 10, 12, 14, 16, 17, 19$ of the permuted square matrix, while the columns of Q become the columns $2, 5, 7, 9, 11, 13, 15, 18$.

In order to implement decomposition via the linear system (4.7), we need the Gram matrices of the B-splines. Note again that in this special case, also these Gram matrices differ only in the number of interior columns, while the blocks corresponding to the boundary elements need to be computed only once and can be used for all refinement levels.

The 19×19 inner product matrix is given by $40320\,\langle \boldsymbol{\gamma}, \boldsymbol{\gamma}^T \rangle =$

$$\begin{pmatrix}
720 & 441 & 93 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
441 & 1116 & \frac{1575}{2} & 174 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
93 & \frac{1575}{2} & 1647 & 1132 & \frac{239}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 174 & 1132 & 2416 & 1191 & 120 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{2} & \frac{239}{2} & 1191 & 2416 & 1191 & 120 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 120 & 1191 & 2416 & 1191 & \frac{239}{2} & \frac{3}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 120 & 1191 & 2416 & 1132 & 174 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{239}{2} & 1132 & 1647 & \frac{1575}{2} & 93 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 174 & \frac{1575}{2} & 1116 & 441 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 93 & 441 & 720
\end{pmatrix}$$

References

1. Ahlberg, J. H. and E. N. Nilson, Orthogonality properties of spline functions, J. Math. Anal. Appl. **11** (1965), 321–337.
2. Boor, C. de, Splines as linear combinations of B-splines. A survey, in *Approximation Theory, II,* G. G. Lorentz, C. K. Chui, and L. L. Schumaker (eds), Academic Press, New York, 1976, 1–47.
3. Boor, C. de, K. Höllig, and S. Riemenschneider, *Box Splines,* Springer Verlag, New York, 1993.
4. Boor, C. de, T. Lyche, and L. L. Schumaker, On calculating with B-splines II. Integration, in *Numerische Methoden der Approximationstheorie Vol. 3, ISNM 30,* L. Collatz, G. Meinardus and H. Werner (eds), Birkhäuser Verlag, Basel, 1976, 123–146.
5. Buhmann, M. and C. A. Micchelli, Spline prewavelets for non-uniform knots, Numer. Math. **61** (1992), 455–474.
6. Carnicer, J. M., W. Dahmen, and J. M. Peña, Local decomposition of refinable spaces and wavelets, Appl. Comput. Harmonic Anal. **3** (1996), 127–153.
7. Chui, C., *An Introduction to Wavelets,* Academic Press, Boston, 1992.
8. Chui, C. K. and E. G. Quak, Wavelets on a bounded interval, in *Numerical Methods in Approximation Theory, ISNM 105,* D. Braess, L.L. Schumaker (ed), Birkhäuser, Basel, 1992, 53–75.
9. Chui, C. K. and J. De Villiers, Spline-wavelets with arbitrary knots on a bounded interval: Orthogonal decomposition and computational algorithms, Comm. in Appl. Anal. **4** (1998), 457–486.
10. Cohen, A., I. Daubechies, and P. Vial, Wavelets on the interval and fast wavelet transforms, Appl. Comput. Harmonic Anal. **1** (1993), 54–81.
11. Cohen, E., T. Lyche, and R. Riesenfeld, Discrete B-splines and subdivision techniques in computer-aided geometric design and computer graphics, Comp. Graphics and Image Proc. **14** (1980), 87–111.
12. Dahmen, W., A. Kunoth, and K. Urban, Biorthogonal spline wavelets on the interval—stability and moment conditions, Appl. Comput. Harmonic Anal. **6** (1999), 132–196.
13. Daubechies, I., *Ten Lectures on Wavelets,* CBMS Conf. Series in Appl. Math., Vol. 61, SIAM, Philadelphia, 1992.
14. Daubechies, I., I. Guskov, P. Schröder, and W. Sweldens, Wavelets on irregular point sets, to appear in Phil. Trans. R. Soc. Lon. A., 2000.
15. Floater, M. S. and E. G. Quak, Piecewise linear prewavelets on arbitrary triangulations, Numer. Math. **82** (1999), 221–252.
16. Floater, M. S. and E. G. Quak, Linear independence and stability of piecewise linear prewavelets on arbitrary triangulations, to appear in SIAM. J. Numer. Anal., 2000.
17. Floater, M. S., E. G. Quak, and M. Reimers, Filter bank algorithms for piecewise linear prewavelets on arbitrary triangulations, to appear in J. Comp. and Appl. Math., 2000.
18. Karlin, S., *Total Positivity,* Stanford Univ. Press, Stanford, 1968.

19. Lounsbery, M., T. D. DeRose, and J. Warren, Multiresolution analysis for surfaces of arbitrary topological type, ACM Trans. Graphics **16** (1997), 34–73.
20. Lyche, T. and K. Mørken, Making the Oslo algorithm more efficient, SIAM J. Numer. Anal. **23** (1986), 663–675.
21. Lyche, T. and K. Mørken, A data reduction strategy for splines, IMA J. Numer. Anal. **8** (1988), 185–208.
22. Lyche, T. and K. Mørken, Spline-wavelets of minimal support, in *Numerical Methods in Approximation Theory, ISNM 105,* D. Braess, L.L. Schumaker (ed), Birkhäuser, Basel, 1992, 177–194.
23. Lyche, T. and L. L. Schumaker, L-spline wavelets, in *Wavelets: Theory, Algorithms, and Applications,* C. Chui, L. Montefusco, and L. Puccio (eds), Academic Press, New York, 1994, 197–212.
24. Lyche, T. and L. L. Schumaker, A multiresolution tensor spline method for fitting functions on the sphere, preprint, 1998.
25. Meyer, Y., Ondelettes sur l'interval, Rev. Mat. Iberoamericana (1992), 115–133.
26. Mørken, K., Some identities for products and degree raising of splines, Constr. Approx. **7** (1991), 195–208.
27. Phillips, J. L. and R. J. Hanson, Gauss quadrature rules with B-spline weight functions, Math. Comp. **28** (1974), 666.
28. Quak, E. G. and N. Weyrich, Decomposition and reconstruction algorithms for spline wavelets on a bounded interval, Appl. Comput. Harmonic Anal. **1** (1994), 217–231.
29. Quak, E. G. and N. Weyrich, Decomposition and reconstruction algorithms for bivariate spline wavelets on the unit square, in *Wavelets, Images, and Surface Fitting,* P. J. Laurent, A. Le Méhauté, and L. L. Schumaker (eds), A. K. Peters, Boston, 1994, 419–428.
30. Quak, E. G. and N. Weyrich, Algorithms for spline wavelet packets on an interval, BIT **37** (1997), 76–95.
31. Schumaker, L. L., *Spline Functions: Basic Theory,* Wiley–Interscience, New York, 1981.
32. Stevenson, R., Piecewise linear (pre-)wavelets on non-uniform meshes, in *Multigrid Methods V,* W. Hackbusch and G. Wittum (eds), Springer, Berlin, 1998, 306–319.
33. Strela, V., Multiwavelets: Theory and applications, Ph. D. thesis, MIT, 1996.
34. Sweldens, W. and P. Schröder, Building your own wavelets at home, in Wavelets in Computer Graphics, ACM SIGGRAPH Course notes, 1996.