

# On the Sup-norm Condition Number of the Multivariate Triangular Bernstein Basis.

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**Abstract.** We give an upper bound for the  $L^\infty$  condition number of the triangular Bernstein basis for polynomials of total degree at most  $n$  in  $s$  variables. The upper bound grows like  $(s+1)^n$  when  $n$  tends to infinity. Moreover the upper bound is independent of  $s$  for  $s \geq n-1$ .

## 1. Introduction

In this paper we estimate the size of the coefficients of a polynomial  $f$  of total degree  $n$  in  $s$  variables when it is represented using the triangular Bernstein basis. This basis has gained increasing popularity mainly through work in Computer Aided Geometric Design [2]. For similar estimates for univariate B-splines see [1,3,4]. We consider only estimates in the  $L^\infty$  norm in this paper. The general  $L^p$  case together with sharpness of the estimates will be published elsewhere.

The condition number of a basis can be defined quite generally.

**Definition 1.1.** A basis  $(\phi_j)$  of a normed linear space is said to be stable with respect to a vector norm if there are constants  $K_1$  and  $K_2$  such that for all coefficients  $(c_j)$

$$K_1^{-1} \|(c_j)\| \leq \left\| \sum_j c_j \phi_j \right\| \leq K_2 \|(c_j)\|. \quad (1)$$

(For simplicity we use the same symbol  $\| \cdot \|$  for the norm in the vector space and the vector norm.) The number

$$\kappa = K_1 K_2$$

with  $K_1$  and  $K_2$  as small as possible is called the condition number of  $(\phi_j)$  with respect to  $\| \cdot \|$ .

Such condition numbers give an upper bound for how much an error in coefficients can be magnified in function values. Indeed, if  $f = \sum_j c_j \phi_j \neq 0$  and

$g = \sum_j d_j \phi_j$  then it follows immediately from (1) that

$$\frac{\|f - g\|}{\|f\|} \leq \kappa \frac{\|c - d\|}{\|c\|}.$$

Many other applications are given in [1] and it is interesting to have estimates for the size of  $\kappa$ .

The contents of this paper is as follows. We recall the definition of the Bernstein basis in Section 2. There we also transform the problem from a simplex to a cube. This makes it possible to analyze the problem in a tensor product fashion. The univariate case was considered in [3]. We extend these results in Section 3. In Section 4 and 5 we consider the multivariate case.

We use standard multi-index notation. Thus for tuples  $\alpha = (\alpha_1, \dots, \alpha_s)$  and  $\mathbf{i} = (i_1, \dots, i_s)$  we let  $|\mathbf{i}| = i_1 + \dots + i_s$ ,  $\mathbf{i}! = i_1! i_2! \dots i_s!$ , and  $\mathbf{i}^\alpha = (i_1^{\alpha_1}, i_2^{\alpha_2}, \dots, i_s^{\alpha_s})$ . Unless otherwise stated the indices in a sum will be nonnegative. Thus if we sum in the order  $\alpha_s, \alpha_{s-1}, \dots, \alpha_1$  then

$$\sum_{|(\alpha_1, \dots, \alpha_s)| \leq n} = \sum_{\alpha_1=0}^n \sum_{\alpha_2=0}^{n-\alpha_1} \sum_{\alpha_3=0}^{n-\alpha_1-\alpha_2} \dots \sum_{\alpha_s=0}^{n-\alpha_1-\dots-\alpha_{s-1}}.$$

The sums  $\sum_{|(i_1, \dots, i_s)| \leq n}$  and  $\sum_{|(\alpha_1, \dots, \alpha_{s+1})|=n}$  will both contain  $\binom{n+s}{s}$  terms. We denote by  $\|\mathbf{c}\|_\infty$  and  $\|f\|_{L^\infty(\Omega)}$  the usual sup-norms of vectors and functions defined on a set  $\Omega$ , respectively. The convex hull of  $m$  points  $v_1, \dots, v_m$  is denoted  $\langle v_1, \dots, v_m \rangle$ . For any  $x \in \mathbb{R}$  the ‘‘floor’’ function  $[x]$  is the unique integer  $n$  so that  $n \leq x < n + 1$ .

## 2. The Bernstein Basis

For the vector space

$$P_n(\mathbb{R}^s) = \{p(\mathbf{x}) = \sum_{|\mathbf{i}| \leq n} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} : c_{\mathbf{i}} \in \mathbb{R}\}$$

of polynomials of total degree at most  $n$  in  $s$  variables  $\mathbf{x} = (x_1, \dots, x_s)$  we consider the Bernstein basis

$$\left\{ \frac{n!}{\alpha!} \lambda^\alpha \right\}_{|\alpha|=n}.$$

Here  $\lambda = (\lambda_1, \dots, \lambda_{s+1})$  denotes the barycentric coordinate with respect to a nondegenerate simplex  $\Sigma = \langle \mathbf{v}_1, \dots, \mathbf{v}_{s+1} \rangle$  in  $\mathbb{R}^s$  i.e., the tuple  $\lambda$  corresponding to a point  $\mathbf{x} \in \mathbb{R}^s$  is uniquely given by

$$\sum_{i=1}^{s+1} \lambda_i \mathbf{v}_i = \mathbf{x}, \quad \sum_{i=1}^{s+1} \lambda_i = 1.$$

Since  $\lambda \geq 0$  for each  $\mathbf{x} \in \Sigma$  and  $\sum_{|\alpha|=n} \frac{n!}{\alpha!} \lambda^\alpha = 1$  for any  $\mathbf{x} \in \mathbb{R}^s$  we have  $K_2 = 1$  in (1) so that

$$\kappa_{n,\infty}(\mathbb{R}^s) = \sup_{\mathbf{c} \neq \mathbf{0}} \left\| \mathbf{c} \right\|_\infty / \left\| \sum_{|\alpha|=n} c_\alpha \frac{n!}{\alpha!} \lambda^\alpha \right\|_{L^\infty(\Sigma)}. \quad (2)$$

For our purpose it is convenient to introduce the change of variables (see [6], p.29)

$$\mathbf{x} \rightarrow \mathbf{y} = (y_1, \dots, y_s)$$

given by

$$\begin{aligned} \lambda_1 &= y_1 & &= y_1 \\ \lambda_2 &= y_2(1 - y_1) & &= y_2(1 - \lambda_1) \\ \lambda_3 &= y_3(1 - y_2)(1 - y_1) & &= y_3(1 - \lambda_1 - \lambda_2) \\ &\vdots & &\vdots \\ \lambda_s &= y_s(1 - y_{s-1}) \cdots (1 - y_1) & &= y_s(1 - \lambda_1 - \lambda_2 - \cdots - \lambda_{s-1}). \end{aligned}$$

This transformation maps  $\Sigma$  onto the  $s$ -dimensional unit cube  $[0, 1]^s$ . Since

$$\lambda_{s+1} = 1 - \lambda_1 - \lambda_2 - \cdots - \lambda_s = (1 - y_s)(1 - y_{s-1}) \cdots (1 - y_1)$$

we obtain

$$\lambda^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_s^{\alpha_s} (1 - y_1)^{n - \alpha_1} (1 - y_2)^{n - \alpha_1 - \alpha_2} (1 - y_s)^{n - \alpha_1 - \cdots - \alpha_s}.$$

Combining this equation with the relation

$$\frac{n!}{\alpha!} = \binom{n}{\alpha_1} \binom{n - \alpha_1}{\alpha_2} \binom{n - \alpha_1 - \alpha_2}{\alpha_3} \cdots \binom{n - \alpha_1 - \cdots - \alpha_{s-1}}{\alpha_s}$$

we see that

$$\frac{n!}{\alpha!} \lambda^\alpha = B_{\alpha_1}^n(y_1) B_{\alpha_2}^{n - \alpha_1}(y_2) \cdots B_{\alpha_s}^{n - \alpha_1 - \cdots - \alpha_{s-1}}(y_s), \quad (3)$$

where

$$B_k^n(x) = \binom{n}{k} x^k (1 - x)^{n - k}, \quad k = 0, 1, \dots, n \quad x \in \mathbb{R} \quad (4)$$

are the usual univariate Bernstein basis polynomials. Thus every polynomial in  $P_n(\mathbb{R}^s)$  can be written in tensor product manner as follows

$$f = \sum_{|\alpha|=n} c_\alpha \frac{n!}{\alpha!} \lambda^\alpha = \sum_{\alpha_1=0}^n B_{\alpha_1}^n(y_1) \left[ \sum_{\alpha_2=0}^{n - \alpha_1} B_{\alpha_2}^{n - \alpha_1}(y_2) \cdots \sum_{\alpha_s=0}^{n - \alpha_1 - \cdots - \alpha_{s-1}} c_\alpha B_{\alpha_s}^{n - \alpha_1 - \cdots - \alpha_{s-1}}(y_s) \right]. \quad (5)$$

### 3. The Univariate Case

When  $s = 1$  then (5) takes the form  $f = \sum_{j=0}^n c_j B_j^n$  where  $B_j^n$  is the univariate Bernstein basis polynomial given by (4) and  $\mathbf{c} = (c_0, \dots, c_n)$  is called the BB-coefficient vector of  $f$ . Thus (1) takes the form

$$\kappa_{n,\infty}(\mathbb{R}^1) = \sup_{\mathbf{c} \neq 0} \|\mathbf{c}\|_{\infty} / \left\| \sum_{j=0}^n c_j B_j^n \right\|_{L^{\infty}[0,1]}.$$

The following Lemma shows that

$$\kappa_{n,\infty}(\mathbb{R}^1) = \max_{0 \leq i \leq n} \gamma_{i,n}. \quad (6)$$

where  $(-1)^{n-i} \gamma_{i,n}$  for  $i = 0, \dots, n$  are the BB-coefficients of the shifted Chebyshev polynomial

$$\hat{T}_n(x) := T_n(2x - 1) = \sum_{j=0}^n (-1)^{n-j} \gamma_{j,n} B_j^n(x).$$

It is well known that the  $\gamma_{i,n}$  are given by  $\gamma_{0,n} = \gamma_{n,n} = 1$  for  $n \geq 0$  and

$$\gamma_{i,n} = \frac{(2n-1)(2n-3) \cdots (2n-2i+1)}{1 \cdot 3 \cdots (2i-1)}, \quad i = 1, \dots, n-1, \quad n \geq 2. \quad (7)$$

**Lemma 3.1.** *For any  $(c_j)$  we have*

$$|c_i| \leq \gamma_{i,n} \left\| \sum_{j=0}^n c_j B_j^n \right\|_{L^{\infty}[0,1]}, \quad i = 0, 1, \dots, n, \quad (8)$$

where the  $\gamma_{i,n}$  are given by (7).

**Proof:** Fix  $0 \leq i \leq n$ . We first show that

$$\inf_{(c_j)} \left\{ \left\| B_i^n - \sum_{j \neq i} c_j B_j^n \right\|_{L^{\infty}[0,1]} \right\} = \|\hat{T}_n / \gamma_{i,n}\|_{L^{\infty}[0,1]} \quad (9)$$

Suppose  $\|V\|_{L^{\infty}[0,1]} < \|\hat{T}_n / \gamma_{i,n}\|_{L^{\infty}[0,1]}$  for some  $V = B_i^n - \sum_{j \neq i} d_j B_j^n$ . Then  $W = V - \hat{T}_n / \gamma_{i,n}$  would change sign at the  $n$  extrema in  $[0, 1]$  of  $\hat{T}_n$ . But this would imply that  $W = 0$ , since  $W \in \text{span}(B_j^n)_{j \neq i}$  and this set forms an order complete weak Chebyshev system on  $[0, 1]$ . (See Theorem 4.65, the remark on p.170 and Theorem 2.42 in [5]). This contradiction establishes (9). From (9) for any nonzero  $c_i$

$$\begin{aligned} \left\| \sum_{j=0}^n c_j B_j^n \right\|_{L^{\infty}[0,1]} &= |c_i| \left\| B_i^n - \sum_{j \neq i} \frac{c_j}{c_i} B_j^n \right\|_{L^{\infty}[0,1]} \\ &\geq |c_i| \|\hat{T}_n / \gamma_{i,n}\|_{L^{\infty}[0,1]} = \frac{|c_i|}{\gamma_{i,n}} \end{aligned}$$

from which the desired estimate follows. ■

To compute  $\max_{0 \leq i \leq n} \gamma_{i,n}$  we observe that the  $\gamma$ 's in (7) satisfy the recurrence relation

$$\gamma_{i,n} = \frac{2(n-i)+1}{2i-1} \gamma_{i-1,n}, \quad i = 1, \dots, n. \quad (10)$$

It follows that

$$\gamma_{0,n} < \dots < \gamma_{m,n} \geq \gamma_{m+1,n} > \dots > \gamma_{n,n}, \quad \text{with } m = \left\lfloor \frac{n}{2} \right\rfloor,$$

and with  $n = 2m + k$  for  $k \in \{0, 1\}$  we find

$$\kappa_{n,\infty}(\mathbb{R}^1) = \gamma_{m,n} = \frac{(2n-1)(2n-3) \cdots (2m+3)(2m+1)^{1-k}}{1 \cdot 3 \cdots (2m-1)}. \quad (11)$$

In [3] the following asymptotic bound was found

$$\left(1 - \frac{1}{n}\right) 2^{n-1/2} \leq \kappa_{n,\infty}(\mathbb{R}^1) \leq \left(1 + \frac{1}{n}\right) 2^{n-1/2}, \quad n \geq 1. \quad (12)$$

In the remaining part of the paper we extend (11) and (12) to an upper bound for  $\kappa_{n,\infty}(\mathbb{R}^s)$  for  $s > 1$ .

#### 4. An Upper Bound for $\kappa_{n,\infty}(\mathbb{R}^s)$

We consider now the case  $s \geq 2$ . We first prove a Lemma

**Lemma 4.1.** *For  $n, s \geq 1$  we have*

$$\kappa_{n,\infty}(\mathbb{R}^s) \leq K_n(\mathbb{R}^s), \quad (13)$$

where  $K_n(\mathbb{R}^1) := \kappa_{n,\infty}(\mathbb{R}^1)$  and for  $s \geq 2$

$$K_n(\mathbb{R}^s) := \max_{0 \leq i \leq n} \gamma_{i,n} K_{n-i}(\mathbb{R}^{s-1}). \quad (14)$$

Here  $\gamma_{i,n}$  is given by (7).

**Proof:** This follows from (5) by repeated application of Lemma 3.1. In order to explain the main idea we consider first the case  $s = 2$ . In this case (5) takes the form

$$\begin{aligned} f &= \sum_{i=0}^n \sum_{k=0}^{n-i} c_{i,k,n-i-k} \frac{n!}{i!k!(n-i-k)!} \lambda_1^i \lambda_2^k \lambda_3^{n-i-k} \\ &= \sum_{i=0}^n \left[ \sum_{k=0}^{n-i} c_{i,k,n-i-k} B_k^{n-i}(y_2) \right] B_i^n(y_1). \end{aligned}$$

Using Lemma 3.1 first on the inner sum and then on the outer sum we obtain

$$\begin{aligned} |c_{i,j,n-i-j}| &\leq \gamma_{j,n-i} \left\| \sum_{k=0}^{n-i} c_{i,k,n-i-k} B_k^{n-i} \right\|_{L^\infty[0,1]} \\ &\leq \gamma_{j,n-i} \gamma_{i,n} \|f\|_{L^\infty[\Sigma]}. \end{aligned}$$

Thus

$$\kappa_{n,\infty}(\mathbb{R}^2) \leq \max_{0 \leq i \leq n} \{ \gamma_{i,n} \max_{0 \leq j \leq n-i} \gamma_{j,n-i} \} = \max_{0 \leq i \leq n} \gamma_{i,n} K_{n-i}(\mathbb{R}^1).$$

This proves (14) for  $s = 2$ . For arbitrary  $s$  a similar argument shows that

$$|c_\alpha| \leq \max_{\mathbf{i} \in J_s^n} \prod_{k=1}^s \gamma_{i_k, n-i_1-\dots-i_{k-1}} \quad \text{for } |\alpha| = n,$$

where  $J_s^n = \{(i_1, \dots, i_s) : 0 \leq i_k \leq n - i_1 - \dots - i_{k-1}, \quad k = 1, \dots, s\}$ . Since

$$\max_{\mathbf{i} \in J_s^n} \prod_{k=1}^s \gamma_{i_k, n-i_1-\dots-i_{k-1}} = \max_{0 \leq i_1 \leq n} \gamma_{i_1, n} \left[ \max_{(i_2, \dots, i_s) \in J_{s-1}^{n-i_1}} \prod_{k=2}^s \gamma_{i_k, n-i_1-\dots-i_{k-1}} \right]$$

(14) follows by induction. ■

The constant  $K_n(\mathbb{R}^s)$  can be computed exactly.

**Theorem 4.2.** For positive integers  $n$  and  $s$

$$\kappa_{n,\infty}(\mathbb{R}^s) \leq K_n(\mathbb{R}^s) = \frac{(2n-1)(2n-3)\cdots(2m+3)(2m+1)^{1-k}}{1^s \cdot 3^s \cdots (2m-1)^s}, \quad (15)$$

where

$$m = \left\lfloor \frac{n}{s+1} \right\rfloor, \quad \text{and } k = n - (s+1)m. \quad (16)$$

Moreover, we have the alternative representations

$$K_n(\mathbb{R}^s) = \frac{(2n)!}{n!} \left( \frac{m!}{(2m)!} \right)^{s+1} (4m+2)^{-k} = \frac{\pi^{s/2} \Gamma(n+1/2)}{(m+1/2)^k \Gamma(m+1/2)^{s+1}}, \quad (17)$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad z > 0,$$

is the usual Gamma function.

**Proof:** By an elementary calculation it is easy to see that (15) and the leftmost formula in (17) define the same number for all  $n, s \geq 1$ . Note that empty products

are defined to be one so that the denominator in (15) is equal to one for  $m = 0$ . For the rightmost formula it suffices to recall the relation (Cf. [7])

$$1 \cdot 3 \cdots (2n - 1) = 2^n \pi^{-1/2} \Gamma(n + 1/2), \quad (18)$$

valid for any nonnegative integer  $n$ .

We shall prove (15) using induction on  $s$ . By (11) we see that (15) holds for  $s = 1$  and all  $n \geq 1$ . Suppose now  $s \geq 2$  and  $n \geq 1$ . By Lemma 4.1

$$K_n(\mathbb{R}^s) = \max_{0 \leq i \leq n} K_{i,n}(\mathbb{R}^s), \quad \text{where} \quad K_{i,n}(\mathbb{R}^s) = \gamma_{i,n} K_{n-i}(\mathbb{R}^{s-1}). \quad (19)$$

Inserting (15) for  $s - 1$  and (7) into (19) we find the explicit formula

$$K_{i,n}(\mathbb{R}^s) = \frac{(2n - 1)(2n - 3) \cdots (2l_i + 3)(2l_i + 1)^{1-j_i}}{1 \cdot 3 \cdots (2i - 1) \cdot 1^{s-1} \cdot 3^{s-1} \cdots (2l_i - 1)^{s-1}}, \quad (20)$$

where  $l_i = \lfloor \frac{n-i}{s} \rfloor$  and  $j_i = n - i - sl_i$ . To determine the  $i$  which gives the max of this expression we show that  $K_{i,n}(\mathbb{R}^s)$  satisfies the recurrence relation

$$K_{i,n}(\mathbb{R}^s) = \frac{2l_i + 1}{2i - 1} K_{i-1,n}(\mathbb{R}^s), \quad i = 1, \dots, n. \quad (21)$$

This is clear from (20) if  $0 \leq j_i \leq s - 2$ . For then  $l_{i-1} = l_i$  and  $j_{i-1} = j_i + 1$ . But it also holds for the remaining case  $j_i = s - 1$ . In this case

$$l_{i-1} = \left\lfloor \frac{n - i + 1}{s} \right\rfloor = \left\lfloor \frac{j_i + sl_i + 1}{s} \right\rfloor = l_i + 1,$$

and since  $j_{i-1} = 0$  we find

$$K_{i-1,n}(\mathbb{R}^s) = \frac{(2n - 1)(2n - 3) \cdots (2l_i + 3)}{1 \cdot 3 \cdots (2i - 3) \cdot 1^{s-1} \cdot 3^{s-1} \cdots (2l_i + 1)^{s-1}}.$$

Thus comparing this with (20) for  $j_i = s - 1$  we see that (21) also holds in this case.

From (21) it follows that

$$K_{0,n} < \cdots < K_{m,n} \geq K_{m+1,n} > \cdots > K_{n,n}.$$

Thus  $K_n(\mathbb{R}^s) = K_{m,n}(\mathbb{R}^s)$  and computing this value from (20) we see that (15) also holds for  $s$ . ■

**Remark.** The above proof also holds for the case  $l_i = 0$  if we interpret the product  $1 \cdot 3 \cdots (2i - 1)$  as 1 if  $i = 0$ . In particular we obtain from (17)

$$K_n(\mathbb{R}^s) = 1 \cdot 3 \cdot 5 \cdots (2n - 1), \quad \text{for} \quad s \geq n - 1. \quad (22)$$

Thus  $K_n(\mathbb{R}^s)$  becomes independent of the space dimension ,e.g. in the cubic case it is the same for all  $s \geq 2$ . This is quite remarkable and recommends the Bernstein-Bézier basis with low degree for work in highly multidimensional problems.

### 5. Asymptotic formulae

To derive asymptotic formulae for the constant  $K_n(\mathbb{R}^s)$  we find it convenient to use the Gamma function representation of  $K_n(\mathbb{R}^s)$ . There is a wealth of formulas for this function see *i.e.*, the classical book [7].

The following theorem generalizes (12) and shows that the number  $2^{-s/2}(s+1)^n$  is a good estimate for  $K_n(\mathbb{R}^s)$  when  $n$  is large compared to  $s$ .

**Theorem 5.1.** *For  $n, s \geq 1$*

$$K_n(\mathbb{R}^s) = 2^{-s/2}(s+1)^n(1+r_{n,s}), \quad \text{where } r_{n,s} \sim \frac{s^2}{n}, \quad n \rightarrow \infty. \quad (23)$$

More precisely, for  $s = 2$  and  $s = 3$  we have

$$\frac{1}{2}3^n\left(1 - \frac{1}{n}\right) \leq K_n(\mathbb{R}^2) \leq \frac{1}{2}3^n\left(1 + \frac{1}{n}\right), \quad n \geq 1, \quad (24)$$

and

$$2^{-3/2}4^n\left(1 - \frac{2}{n}\right) \leq K_n(\mathbb{R}^3) \leq 2^{-3/2}4^n\left(1 + \frac{2}{n}\right), \quad n \geq 1. \quad (25)$$

**Proof:** Taking logarithms in (17) we have

$$\log K_n(\mathbb{R}^s) = \frac{s}{2} \log \pi + \log \Gamma\left(n + \frac{1}{2}\right) - (s+1) \log \Gamma\left(m + \frac{1}{2}\right) - k \log\left(m + \frac{1}{2}\right), \quad (26)$$

where  $m$  and  $k$  are such that

$$n = (s+1)m + k, \quad \text{with } 0 \leq k \leq s. \quad (27)$$

From [7, p 252-253] we recall Stirlings asymptotic formula for the logarithm of the Gamma function

$$\log \Gamma(z) = (z - 1/2) \log z - z + \frac{1}{2} \log 2\pi + \phi(z), \quad (28)$$

where

$$\phi(z) = \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \dots$$

Since the series is alternating it is shown in [7 p 253] that for  $z > 0$  the upper bound  $1/(12z)$  holds for  $\phi(z)$ . Similarly we also have a lower bound. For  $z \geq 1/2$  the bounds take the form

$$\frac{1}{15z} \leq \frac{1}{12z} - \frac{1}{360z^3} < \phi(z) < \frac{1}{12z}. \quad (29)$$

Inserting (28) in (26) and using elementary properties of logarithms it follows after some calculation that

$$K_n(\mathbb{R}^s) = 2^{-s/2}(s+1)^n E_{n,s}, \quad (30)$$

where

$$\log E_{n,s} = \psi\left(\frac{s}{2} - k, n + \frac{1}{2}\right) + \phi\left(n + \frac{1}{2}\right) - (s+1)\phi\left(m + \frac{1}{2}\right), \quad (31)$$

and for any  $x, z$  such that  $1 + x/z > 0$

$$\psi(x, z) = x - (z - 1/2) \log\left(1 + \frac{x}{z}\right). \quad (32)$$

For  $\psi(x, z)$  we have for  $-1 < x/z < 1$  the series expansion

$$\psi(x, z) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k (x + \frac{k+1}{2k})}{(k+1)z^k}. \quad (33)$$

To give upper and lower bounds for  $E_{n,s}$  we first show that for fixed  $m \geq 0$

$$E_{m(s+1)+[\frac{s+1}{2}],s} \leq E_{m(s+1)+k,s} \leq E_{m(s+1),s}, \quad k = 0, \dots, s. \quad (34)$$

Here we define  $K_0(\mathbb{R}^s) = E_{0,s} = 1$  for all  $s \geq 1$ . To show this we combine (30) and (15) to obtain for  $n = m(s+1), \dots, m(s+1) + s$

$$\frac{E_{n+1,s}}{E_{n,s}} = \frac{K_{n+1}(\mathbb{R}^s)}{(s+1)K_n(\mathbb{R}^s)} = \frac{n+1/2}{(s+1)(m+1/2)}. \quad (35)$$

Thus  $E_{n+1,s} \leq E_{n,s}$  for  $m(s+1) \leq n \leq m(s+1) + s/2$  and  $E_{n+1,s} \geq E_{n,s}$  for  $m(s+1) + s/2 \leq n \leq m(s+1) + s$ . This shows the lower bound in (34) and also we see that the maximum of  $E_{n,s}$  with  $n$  in the range  $n = m(s+1), \dots, m(s+1) + s$  must occur at either  $n = m(s+1)$  or at  $n = m(s+1) + s$ . From (35) it can be seen that

$$\frac{E_{m(s+1)+s,s}}{E_{m(s+1),s}} = \frac{(x - (s-1/2))(x - (s-3/2)) \cdots (x + (s-3/2))(x + (s-1/2))}{(x+1/2)(x+1/2) \cdots (x+1/2)(x+1/2)}$$

where  $x = (s+1)m + 1/2$ . It follows that  $E_{m(s+1)+s,s}/E_{m(s+1),s} < 1$  for all  $s \geq 1$  and all  $m \geq 0$  so the upper bound in (34) follows.

In the following we do not estimate  $E_{n,s}$  in (34) for any  $k$  but only in the interesting cases for the upper and lower bounds. So suppose now for the upper bound of  $E_{n,s}$  that  $n = m(s+1)$  for some  $m \geq 1$ .

For  $x > 0$  the series (33) is alternating and taking only the first term in the series we obtain

$$\psi(x, z) < \frac{x(x+1)}{2z}, \quad (36)$$

valid for  $x, z > 0$ . Using (36), (29), and (31) we obtain with  $z = n + 1/2$

$$\begin{aligned} \log E_{n,s} &= \psi\left(\frac{s}{2}, z\right) + \phi(z) - (s+1)\phi\left(m + \frac{1}{2}\right) \\ &\leq \frac{s^2 + 2s}{8z} + \frac{1}{12z} - \frac{(s+1)^2}{15(z+s/2)} \\ &< \frac{(s+1)^2}{8z} - \frac{(s+1)^2}{16(z+s/2)} = \frac{(s+1)^2}{16z} \frac{z+s}{(z+s/2)}. \end{aligned}$$

This proves the upper bound

$$K_n(\mathbb{R}^s) \leq 2^{-s/2}(s+1)^n \exp\left(\frac{1}{16} \frac{(s+1)^2}{n+1/2} \cdot \frac{n+1/2+s}{n+1/2+s/2}\right), \quad n \geq 1. \quad (37)$$

We now show the lower bound

$$K_n(\mathbb{R}^s) \geq 2^{-s/2}(s+1)^n \exp\left(-\frac{(s+1)^2+1}{12n}\right), \quad n \geq 1. \quad (38)$$

To show (38) there are two cases. First, if  $s$  is even then we need to consider  $n$  of the form  $n = m(s+1) + s/2$  for some  $m \geq 0$ . From (31) and (29) we obtain with  $z = n + 1/2$

$$\begin{aligned} \log E_{n,s} &= \psi(0, z) + \phi\left(n + \frac{1}{2}\right) - (s+1)\phi\left(m + \frac{1}{2}\right) \\ &> 0 + \frac{1}{15z} - \frac{(s+1)^2}{12z} > -\frac{(s+1)^2}{12z}. \end{aligned} \quad (39)$$

Next if  $s$  is odd then the  $n$  which gives the lower bound is of the form  $n = m(s+1) + s/2 + 1/2$  for some  $m \geq 0$ . From (33) we obtain

$$\psi\left(-\frac{1}{2}, z\right) = -\sum_{k=1}^{\infty} \frac{1}{2^{k+1}k(k+1)z^k} > -\frac{1}{4} \sum_{k=1}^{\infty} (2z)^{-k} = -\frac{1}{8(z-1/2)}.$$

Therefore, from (31) and (29) we now obtain with  $z = n + 1/2$  and  $z > 4/3$

$$\begin{aligned} \log E_{n,s} &= \psi(-1/2, z) + \phi\left(n + \frac{1}{2}\right) - (s+1)\phi\left(m + \frac{1}{2}\right) \\ &> -\frac{1}{8(z-1/2)} + \frac{1}{15z} - \frac{(s+1)^2}{12(z-1/2)} \end{aligned}$$

For  $z > 4/3$  we have  $1/(15z) > 1/(24(z-1/2))$  which gives

$$\log E_{n,s} > -\frac{1}{12(z-1/2)}\left(\frac{12}{8} - \frac{1}{2} + (s+1)^2\right) = -\frac{(s+1)^2+1}{12(z-1/2)}.$$

This bound is smaller than (39) and since  $z - 1/2 = n$  we obtain (38).

Consider next the specific cases  $s = 2$ . Setting  $s = 2$  in the upper bound (37) we find

$$\log E_{n,2} \leq \frac{9}{16(n+1/2)} \frac{n+5/2}{n+3/2} =: x.$$

Since  $0 < x \leq 1/2$  for  $n \geq 2$  we have

$$E_{n,2} \leq e^x \leq 1 + \frac{4}{3}x \leq 1 + \frac{1}{n}$$

and the upper bound in (24) follows for  $n \geq 2$ . But since  $K_1(\mathbb{R}^2) = 1$  (24) also holds for  $n = 1$ . The lower bound follows immediately from (38) and the inequality  $e^x \geq 1 - x$  valid for all  $x$ .

Consider finally the case  $s = 3$ . With  $x = (n + 7/2)/(n + 1/2)(n + 2)$  we have  $0 < x \leq 1/2$  for  $n \geq 3$  and for these  $n$  we obtain

$$E_{n,3} \leq e^x \leq 1 + \frac{4}{3}x \leq 1 + \frac{2}{n}.$$

It is shown directly that the same bound is valid for  $n \leq 2$ . Thus the upper bound follows. For the lower bound we argue as for  $s = 2$ . This completes the proof. ■

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