On the existence of piecewise exponential B-splines

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Dedicated to Professor Charles A. Micchelli for his 60th birthday

Abstract. We give necessary and sufficient conditions for the total positivity of certain connection matrices arising in piecewise exponential spline spaces. These total positivity conditions are sufficient for existence of B-splines in such spaces, but they are far from being necessary. We give a necessary and sufficient condition for existence of B-splines in the case of piecewise exponential spline spaces with only two differential operators, which eventually leads to a necessary condition for any piecewise exponential spline spaces.

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§1. Introduction

When computing smooth curves and surfaces one often uses splines represented as B-splines. This leads to stable and robust algorithms and approximation problems which are easy to solve, *i.e.*, involving banded linear system. In CAGD we would in addition like representations that have good shape controlling properties and for this several generalizations of polynomial splines have been proposed. Special cases of Chebyshevian splines and L-splines are two such generalizations and these spline spaces have a basis of B-splines, see [14,16].

One example of such splines are the classical splines in tension also called hyperbolic tension splines. On each polynomial piece these functions belong to the null space of the fourth order differential operator $D^2(D^2-\rho^2)$. By increasing the tension parameter ρ we can construct C^2-interpolants to given data that converges uniformly to the piecewise linear interpolant. In practice we would like more local control and use different parameters ρ on different intervals. This leads to spaces of piecewise hyperbolic tension splines and more generally piecewise Chebyshevian splines and piecewise L-splines. For such spaces it is much harder to show existence of a B-spline basis. It was shown by Barry [1] that for piecewise Chebyshevian splines a sufficient condition for the existence of B-splines is that certain matrices connecting derivatives on each side of a knot are totally positive.

In this paper we study splines with sections in null spaces of linear differential operators with constant coefficients and with only real roots. The orders of the differential operators are constant, but the roots can vary from interval to interval. Even for this special case it is not easy to give necessary and sufficient conditions for the total positivity of the corresponding connection matrices.

The contents of this paper is as follows. In Section 2 we use blossoming to give the necessary background for piecewise Chebyshevian splines. In particular we recall the fundamental result which will be used throughout the paper, namely the fact that existence of B-spline bases is equivalent to existence of blossoms. As an example we show that B-splines always exist in the space of piecewise hyperbolic tension splines. In Section 3 we give necessary and sufficient conditions for total positivity of connection matrices corresponding to the piecewise exponential spline spaces. We give an example to show that, if these conditions are sufficient for the existence of piecewise exponential B-splines, they are not necessary. As a step towards finding necessary conditions we consider in Section 4 the special case of only two differential operators L_1 and L_2. We use L_1 on all segments to the left of some knot t_l and L_2 on all segments to the right of t_l. In this particular case we are able to give necessary and sufficient conditions. In Section 5, we prove that the previous result eventually leads to necessary conditions for existence of piecewise exponential B-spline bases independently of the knots. The latter conditions are sufficient in the 4-dimensional (cubic) case with simple knots.

§2. Piecewise Chebyshevian splines

2.1. The general setting

Consider a bi-infinite sequence $\mathcal{T} = (t_\ell)_{\ell\in\mathbb{Z}}$, with $t_\ell < t_{\ell+1}$ and $\lim_{\ell\to\infty} t_\ell = \infty$. For each $\ell \in \mathbb{Z}$, we assume that $\mathcal{E}_\ell$ is a linear subspace of $C^\infty([t_\ell, t_{\ell+1}])$ satisfying the following properties:

- $\mathcal{E}_\ell$ contains the constants,
- $D\mathcal{E}_\ell := \{DU \mid U \in \mathcal{E}_\ell\}$ is an n-dimensional extended Chebyshev space on $[t_\ell, t_{\ell+1}]$ (*i.e.*, any nonzero element of $D\mathcal{E}_\ell$ vanishes at most $n-1$ times in $[t_\ell, t_{\ell+1}]$, counting multiplicities). This implies that the space $\mathcal{E}_\ell$ itself is an $(n+1)$-dimensional extended Chebyshev space on $[t_\ell, t_{\ell+1}]$).

Let $\mathcal{E}$ denote the space of all C^n functions on $\mathbb{R}$ the restrictions of which to $[t_\ell, t_{\ell+1}]$ belong to $\mathcal{E}_\ell$, $\ell \in \mathbb{Z}$. Due to $\mathcal{E}_\ell$ being an $(n+1)$-dimensional extended Chebyshev space on $[t_\ell, t_{\ell+1}]$, a function $U \in \mathcal{E}_\ell$ is completely determined by the knowledge of $U(t_\ell), \dots, U^{(n)}(t_\ell)$. The C^n requirement therefore implies that the space $\mathcal{E}$ is $(n+1)$-dimensional. Moreover, the space $\mathcal{E}$ contains the constants and it is a *W-space* on $\mathbb{R}$, in the sense that the Wronskian of any basis of $\mathcal{E}$ never vanishes on $\mathbb{R}$.

A vector function $\Phi := (\Phi_1, \dots, \Phi_d)^T : \mathbb{R} \to \mathbb{R}^d$ is called an $\mathcal{E}$-function when all its components $\Phi_1, \dots, \Phi_d$ belong to $\mathcal{E}$. Given an $\mathcal{E}$-function Φ, at any point $x \in \mathbb{R}$ we can consider its ith order osculating flat ($i \le n$), that is the affine subspace of $\mathbb{R}^d$ which passes through $\Phi(x)$ and the direction of which is spanned by $\Phi'(x), \dots, \Phi^{(i)}(x)$, *i.e.*, $\mathrm{Osc}_i\,\Phi(x) := \{\Phi(x) + \lambda_1 \Phi'(x) + \cdots + \lambda_i \Phi^{(i)}(x) \mid \lambda_1, \dots, \lambda_i \in \mathbb{R}\}$. Such an $\mathcal{E}$-function Φ is said to be *nondegenerate* if $(\mathbb{1}, \Phi^1, \dots, \Phi^d)$ span the space $\mathcal{E}$. From now on, the notation $\tau^{[\nu]}$ will mean that the point τ is repeated ν times. Choose a nondegenerate $\mathcal{E}$-function Φ. Throughout the paper, for any $i \le n$, we denote by $\mathrm{Osc}_i\Phi(x)$ the ith order osculating flat of Φ at a given point $x \in \mathbb{R}$, that is, the affine flat containing $\Phi(x)$ and with direction spanned by the i vectors $\Phi'(x), \dots, \Phi^{(i)}(x)$. Given an n-tuple $(x_1, \dots, x_n)$, up to a permutation, we can write it as $({\tau_1}^{[\nu_1]}, \dots, {\tau_r}^{[\nu_r]})$, where $\nu_1, \dots, \nu_r$ are positive integers such that $\sum_{i=1}^r \nu_i = n$, and where $\tau_1 < \cdots < \tau_r$. We shall set

$$\{\,\varphi(x_1, \dots, x_n)\,\} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\nu_i}\,\Phi(\tau_i)\ , \tag{2.1}$$

whenever the osculating flats $\mathrm{Osc}_{n-\nu_i}\,\Phi(\tau_i)$, $1 \le i \le r$, intersect at a single point. Let $\mathcal{D}(\varphi)$ denote the domain of definition of the function φ. It is independent of the chosen nondegenerate $\mathcal{E}$-function Φ.

To each t_ℓ, let us allocate a multiplicity $m_\ell \in \{0, \dots, n\}$. Associated with the corresponding knot vector $\mathcal{K} := \big({t_\ell}^{[m_\ell]}\big)_{\ell\in\mathbb{Z}}$, we define the *spline space based on* $\mathcal{E}$ as the space $\mathcal{S}$ of all functions $S : \mathbb{R} \to \mathbb{R}$ such that, for any $\ell \in \mathbb{Z}$,

(i) S is C^{n-m_ℓ} at the knot t_ℓ,

(ii) the restriction of S to $[t_\ell, t_{\ell+1}]$ belongs to $\mathcal{E}_\ell$.

Clearly the spline space $\mathcal{S}$ contains the space $\mathcal{E}$. Given a spline $S = (S^1, \dots, S^d) \in \mathcal{S}^d$, for any $\ell \in \mathbb{Z}$, there exists a unique $\mathcal{E}$-function $F_\ell \in \mathcal{E}^d$ such that $S(t) = F_\ell(t)$ for all $t \in [t_\ell, t_{\ell+1}]$. We shall say that S is *nondegenerate* if each F_ℓ is nondegenerate.

An n-tuple $(x_1,\ldots,x_n)$ is said to be $\mathcal{K}$-*admissible* (in short, admissible) if any t_ℓ satisfying $\mathrm{Min}(x_1,\ldots,x_n) < t_\ell < \mathrm{Max}(x_1,\ldots,x_n)$ appears at least m_ℓ times in the sequence $x_1,\ldots,x_n$. We denote by $\mathcal{A}$ the set of all admissible n-tuples. Select a non degenerate spline $\Sigma \in \mathcal{S}^d$, and denote by Φ_ℓ the (nondegenerate) $\mathcal{E}$-function which coincides with Σ on $[t_\ell, t_{\ell+1}]$. Let $(x_1,\ldots,x_n)$ be admissible, with, up to a permutation, $(x_1,\ldots,x_n) = ({\tau_1}^{[\nu_1]},\ldots,{\tau_r}^{[\nu_r]})$, where $\nu_1,\ldots,\nu_r$ are positive. Then we have (see[10])

$$\bigcap_{i=1}^{r} \mathrm{Osc}_{n-\nu_i}\,\Sigma(\tau_i) = \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\nu_i}\,\Phi_{j(x_1,\ldots,x_n)}(\tau_i)\ , \tag{2.3}$$

where $j(x_1,\ldots,x_n)$ denotes the greatest among all integers ℓ such that $t_\ell \le \mathrm{Min}(x_1,\ldots,x_n)$. In the left-hand side of the latter equality, $\mathrm{Osc}_{n-\nu_1}\,\Sigma(\tau_1)$ is to be understood as $\mathrm{Osc}_{n-\nu_1}\,\Sigma(\tau_1^+)$ in case $\tau_1 = t_j$ and $\nu_1 < m_j$, and $\mathrm{Osc}_{n-\nu_r}\,\Sigma(\tau_r)$ is to be understood as $\mathrm{Osc}_{n-\nu_1}\,\Sigma(\tau_r^-)$ if $\tau_r = t_{j'}$ and $\nu_r < m_{j'}$. With the latter convention, we shall then set

$$\{\sigma(x_1,\ldots,x_n)\} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\nu_i}\,\Sigma(\tau_i) \tag{2.4}$$

provided that the intersection $\bigcap_{i=1}^{r} \mathrm{Osc}_{n-\nu_i}\,\Sigma(\tau_i)$ consists of a single point. Due to (2.3) the domain of definition $\mathcal{D}(\sigma)$ of the function σ is equal to $\mathcal{A} \cap \mathcal{D}(\varphi)$.

Definition 2.1. *We say that blossoms exist in the space $\mathcal{E}$, if $\mathcal{D}(\varphi) = \mathbb{R}^n$, in which case the function φ defined on $\mathbb{R}^n$ by (2.1) is called the blossom of the nondegenerate $\mathcal{E}$-function Φ. We say that blossoms exist in the spline space $\mathcal{S}$ if $\mathcal{D}(\sigma) = \mathcal{A}$, i.e., if $\mathcal{D}(\varphi) \supset \mathcal{A}$, in which case the function σ defined on $\mathcal{A}$ by (2.4) is called the blossom of the nondegenerate spline Σ.*

Any $\mathcal{E}$-function F is the image $h \circ \Phi$ of Φ under a unique affine map h defined on the affine space spanned by the image of Φ. As usual, when $\mathcal{D}(\varphi) = \mathbb{R}^n$, the blossom f of F is then defined as $f := h \circ \varphi$. Similarly when $\mathcal{D}(\varphi) \supset \mathcal{A}$, blossoms of spline functions are defined (on the set $\mathcal{A}$) from σ by means of affine maps. It was proved in [13] that existence of blossoms is equivalent to existence of B-spline bases, in the following sense, which justifies the interest of the blossoming approach to B-splines.

Theorem 2.2. *Blossoms exist in the space $\mathcal{E}$ if and only if there exist a B-spline basis in any spline space based on $\mathcal{E}$. Blossoms exist in the space $\mathcal{S}$ if and only if there exists a B-spline basis in the space $\mathcal{S}$ and in any spline space obtained from $\mathcal{S}$ by insertion of knots.*

Remark 2.3. The expression *B-spline basis* usually denotes a normalized basis consisting of minimally supported splines which are positive on the interior of their supports. However these requirements are not sufficient to clearly identify the B-splines in case of multiple knots. The necessity of an additional requirement on the numbers of zeros at the end points of the supports was pointed out in [12]. The latter *end point property* is important because it ensures the unicity of a possible B-spline basis. For more details, we refer to [12, 13]. Later on, we shall widely use Theorem 2.2, replacing the search for a B-spline basis (this expression implicity including the additional end point property), by the search for blossoms.

2.3. Weight functions and total positivity

In this subsection we shall recall conditions ensuring the existence of blossoms on either space $\mathcal{E}$ or $\mathcal{S}$. Without loss of generality, we can choose the nondegenerate $\mathcal{E}$-function Φ so that $\Phi = (\Phi^1,\ldots,\Phi^n)$, where $(\mathbb{1},\Phi^1,\ldots,\Phi^n)$ is a basis of $\mathcal{E}$. Let us denote by $\langle\,.\,,.\,\rangle$ the usual inner product in $\mathbb{R}^n$, and by $\wedge$ the associated cross product. Because $\mathcal{E}$ is a W-space on $\mathbb{R}$, for any $x \in \mathbb{R}$, the n conditions

$$\langle\, \Phi^\sharp(x), \Phi^{(r)}(x)\,\rangle = \delta_{n,r}\ , \quad r = 1,\ldots,n\ . \tag{2.5}$$

define a unique vector $\Phi^\sharp(x) = (\Phi^\sharp_1(x), \ldots, \Phi^\sharp_n(x))^T \in \mathbb{R}^n$. which we can write as follows:

$$\Phi^\sharp(x) = \frac{\Phi'(x) \wedge \ldots \wedge \Phi^{(n-1)}(x)}{\det(\Phi'(x), \ldots, \Phi^{(n)}(x))} \ . \tag{2.6}$$

This vector gives the direction orthogonal to the osculating hyperplane $\mathrm{Osc}_{n-1}\,\Phi(x)$. From (2.6) we can see that the function $\Phi^\sharp$ is C^∞ on each $[t_\ell, t_{\ell+1}]$. By iterated differentiations of (2.5), it is easy to check that, at a given point $x \in \mathbb{R}$, the i-dimensional linear subspace orthogonal to $\mathrm{Osc}_{n-i}\,\Phi(x)$, $i \le n$, is spanned by $\Phi^\sharp(x), \ldots, \Phi^{\sharp\,(i-1)}(x)$, *i.e.*,

$$\big[\mathrm{span}\big(\Phi'(x), \ldots, \Phi^{(n-i)}(x)\big)\big]^\perp = \big[\mathrm{span}\big(\Phi^\sharp(x), \ldots, \Phi^{\sharp\,(i-1)}(x)\big)\big] \ . \tag{2.7}$$

In equality (2.7) and throughout the paper, when $x = t_\ell$, the sequence $\Phi^\sharp(x), \ldots, \Phi^{\sharp\,(i-1)}(x)$ is to be understood either as $\Phi^\sharp(t_\ell), \Phi^{\sharp\,\prime}(t_\ell{}^-) \ldots, \Phi^{\sharp\,(i-1)}(t_\ell{}^-)$

or as $\Phi^\sharp(t_\ell), \Phi^{\sharp\,\prime}(t_\ell{}^+) \ldots, \Phi^{\sharp\,(i-1)}(t_\ell{}^+)$. With this convention, it results from (2.7) that the intersection $\bigcap_{i=1}^r \mathrm{Osc}_{n-\nu_i}\,\Phi(\tau_i)$ consists of a single point if and only if the linear system

$$\langle\, X, \Phi^{\sharp\,(k)}(\tau_i)\,\rangle = \langle\, \Phi(\tau_i), \Phi^{\sharp\,(k)}(\tau_i)\,\rangle \ , \ 0 \le k \le \nu_i - 1 \ , \ 1 \le i \le r \ ,$$

has a unique solution X (for more details, we refer the reader to [10, Section 2]). We thus eventually obtain the following result, which we shall use in Section 4.

Theorem 2.4. *For given positive integers $\nu_1, \ldots, \nu_r$, with $\sum_{i=1}^r \nu_i = n$, and given pairwise dictinct $\tau_1, \ldots, \tau_r \in \mathbb{R}$, the intersection $\bigcap_{i=1}^r \mathrm{Osc}_{n-\nu_i}\,\Phi(\tau_i)$ consists of a single point if and only if*

the n vectors $\Phi^\sharp(\tau_i), \ldots, \Phi^{\sharp\,(\nu_i-1)}(\tau_i), i = 1, \ldots, r,$ are linearly independent. (2.8)

In particular, the previous result states that blossoms exist in the space $\mathcal{S}$, if and only if condition (2.8) is satisfied whenever the n-tuple $(\tau_1{}^{[\nu_1]}, \ldots, \tau_r{}^{[\nu_r]})$ is admissible, while blossoms exist in the space $\mathcal{E}$, if and only if condition (2.8) is always satisfied. Note that the space $\mathcal{E}^\sharp$ spanned by the components $\Phi^\sharp_1, \ldots, \Phi^\sharp_n$ of $\Phi^\sharp$ is an n-dimensional space of continuous piecewise C^∞ functions (relatively to the sequence $\mathcal{T}$). Requiring condition (2.8) to be satisfied for any n-tuple is equivalent to requiring that any nonzero elements of $\mathcal{E}^\sharp$ has at most $n-1$ zeros in $\mathbb{R}$, counting multiplicities. As we shall recall hereunder, sufficient conditions to bound the number of zeros of elements of $\mathcal{E}^\sharp$ can be obtained by using weight functions and total positivity.

Given a closed bounded interval J, and a subspace $\mathcal{F}$ of $C^\infty(J)$ containing the constants, it is well-known that the space $D\mathcal{F}$ is an n-dimensional extended Chebyshev space on J if and only if there exist n positive weight functions $w_1, \ldots, w_n \in C^\infty(J)$ such that $\mathcal{F} = \mathrm{Ker}L_{n+1}$, where $L_1, \ldots, L_{n+1}$ are the linear differential operators defined on $C^\infty(J)$ by

$$L_1 u := \frac{1}{w_1}\ u' \ , \quad L_k u := \frac{1}{w_k}\ \big(L_{k-1}u\big)' \ , \quad k = 2, \ldots, n+1 \ , \tag{2.9}$$

with $w_{n+1} := \mathbb{1}$. For each $\ell \in \mathbb{Z}$, we can thus write $\mathcal{E}_\ell = \mathrm{Ker}L^\ell_{n+1}$, where the differential operators $L^\ell_1, \ldots, L^\ell_{n+1}$ are defined from positive weight functions $w^\ell_1, \ldots, w^\ell_n \in C^\infty([t_\ell, t_{\ell+1}])$ according to (2.9). For all $u \in C^\infty([t_\ell, t_{\ell+1}])$ and all $x \in [t_\ell, t_{\ell+1}]$, we have

$$\big(L^\ell_1 u(x), \ldots, L^\ell_n u(x)\big)^T = \mathcal{A}_\ell(x)\ .\ \big(u'(x), \ldots, u^{(n)}(x)\big)^T \ , \tag{2.10}$$

where $\mathcal{A}_\ell(x)$ is a lower triangular matrix with diagonal $\big(1/\prod_{1\le i\le k} w_i(x)\big)_{1\le k\le n}$. Using (2.10), the space $\mathcal{E}$ can as well be described as the set of all functions U defined on $\mathbb{R}$, the restrictions of which to $[t_\ell, t_{\ell+1}]$ belong to $\mathcal{E}_\ell$, $\ell \in \mathbb{Z}$, and which satisfy the connection conditions

$$\big(L^\ell_1 U(t_\ell{}^+), \ldots, L^\ell_n U(t_\ell{}^+)\big)^T = \mathcal{M}_\ell\ .\ \big(L^{\ell-1}_1 U(t_\ell{}^-), \ldots, L^{\ell-1}_n U(t_\ell{}^-)\big)^T \ , \quad \ell \in \mathbb{Z} \ , \tag{2.11}$$

where

$$\mathcal{M}_\ell := \mathcal{A}_\ell(t_\ell) \,.\, \mathcal{A}_{\ell-1}(t_\ell)^{-1} \,. \tag{2.12}$$

Theorem 2.5. *If each matrix $\mathcal{M}_\ell$ is totally positive (i.e., if all minors of $\mathcal{M}_\ell$ are nonnegative), then blossoms exist in the space $\mathcal{E}$.*

Proof: The total positivity actually enables us to apply a result of P.J. Barry [1] to bound the number of zeros of nonzero elements of $\mathcal{E}^\sharp$. We refer to [10] for more details. ■

Note that, in order to apply Barry's result, the order of the weight functions is fixed once and for all.

Similarly, the spline space $\mathcal{S}$ can be described as the set of all functions U defined on $\mathbb{R}$, the restrictions of which to $[t_\ell, t_{\ell+1}]$ belong to $\mathcal{E}_\ell$, $\ell \in \mathbb{Z}$, and which satisfy the continuity conditions

$$\left(L_1^\ell S(t_\ell{}^+), \ldots, L_{n-m_\ell}^\ell S(t_\ell{}^+)\right)^T = \widehat{\mathcal{M}}_\ell \,.\, \left(L_1^{\ell-1} S(t_\ell{}^-), \ldots, L_{n-m_\ell}^{\ell-1} S(t_\ell{}^-)\right)^T \,, \quad \ell \in \mathbb{Z} \,, \tag{2.13}$$

where $\widehat{\mathcal{M}}_\ell$ is the $(n-m_\ell)$th order square matrix obtained by suppressing the m_ℓ last rows and columns in $\mathcal{M}_\ell$.

Theorem 2.6. *If each $\widehat{\mathcal{M}}_\ell$ is totally positive, then blossoms exist in the spline space $\mathcal{S}$.*

Proof: For any integer ℓ such that $m_\ell > 0$, let us add m_ℓ rows and columns to $\widehat{\mathcal{M}}_\ell$ so as to obtain a totally positive matrix $\widetilde{\mathcal{M}}_\ell$. Consider the space $\widetilde{\mathcal{E}}$ composed all of functions all functions U defined on $\mathbb{R}$, the restrictions of which to $[t_\ell, t_{\ell+1}]$ belong to $\mathcal{E}_\ell$, $\ell \in \mathbb{Z}$, and which satisfy the continuity conditions

$$\left(L_1^\ell U(t_\ell{}^+), \ldots, L_n^\ell U(t_\ell{}^+)\right)^T = \widetilde{\mathcal{M}}_\ell \,.\, \left(L_1^{\ell-1} U(t_\ell{}^-), \ldots, L_n^{\ell-1} U(t_\ell{}^-)\right)^T \,, \quad \ell \in \mathbb{Z} \,. \tag{2.14}$$

Blossoms exist in the space $\widetilde{\mathcal{E}}$, and therefore in any spline space based on $\widetilde{\mathcal{E}}$, thus in particular in $\mathcal{S}$. ■

2.4. An example: piecewise hyperbolic tension spaces

Suppose that, for each $\ell \in \mathbb{Z}$, the weight functions $w_1^\ell, \ldots, w_n^\ell$ are obtained by restriction to $[t_\ell, t_{\ell+1}]$ of positive functions $w_1, \ldots, w_n$ defined on the whole of $\mathbb{R}$, with $w_i \in C^{n-i}(\mathbb{R})$, $1 \le i \le n$. Existence of blossoms in the space $\mathcal{E}$ directly follows from Theorem 2.5, indeed, all matrices $\mathcal{M}_\ell$ obtained in (2.12) are totally positive since they all are equal to the identity matrix. Hence, existence of B-spline bases in any spline space based on $\mathcal{E}$ is ensured by Theorem 2.2. Note that in this particular case, the space $\mathcal{E}$ is an extended complete Chebyshev subspace of $C^n(\mathbb{R})$ and existence of locally supported bases thus follows from [16].

As we shall see now, this case contains piecewise hyperbolic tension spaces. Given a nonnegative number r, consider the kernel of the differential operator $D^{n-1}(D^2 - r^2 I)$, that is, the polynomial space $\mathcal{P}_n$ of degree n when $r = 0$, and the space spanned by $1, x \ldots, x^{n-2}, \cosh rx, \sinh rx$ otherwise. We shall refer to it as the *hyperbolic tension space* associated with r. It is known that it is an extended Chebyshev space on $\mathbb{R}$. For the rest of the subsection we suppose that each space $\mathcal{E}_\ell$ is the restriction to $[t_\ell, t_{\ell+1}]$ of the hyperbolic tension space associated with some nonnegative number r_ℓ depending on ℓ. We shall then say that the space $\mathcal{E}$ is a *piecewise hyperbolic tension space* and we shall refer to the corresponding spline spaces based on $\mathcal{E}$ as *piecewise hyperbolic tension spline spaces.*

Proposition 2.7. *Blossoms exist in any piecewise hyperbolic tension spaces (and therefore B-spline bases exist in any piecewise hyperbolic tension spline spaces).*

Proof: Suitable weight functions associated with the space $\mathcal{E}_\ell$ can be obtained as follows. Set $w_1^\ell := \cdots := w_{n-2}^\ell := \mathbb{1}$, choose a function w_{n-1}^ℓ in the kernel of the differential operator $D^2 - r_\ell{}^2 I$

so that it is positive on $[t_\ell, t_{\ell+1}]$, and define the last weight function as $w_n^\ell := (w_{n-1}^\ell)^{-2}$. Indeed, the corresponding differential operator L_{n+1}^ℓ is then given by

$$L_{n+1}^\ell u = w_{n-1}^\ell \big[u^{(n+1)} - {r_\ell}^2 u^{(n-1)}\big] \ .$$

We shall now prove that it is possible to choose such a weight function w_{n-1}^ℓ, as the restriction to $[t_\ell, t_{\ell+1}]$ of a given function w_{n-1} which is C^1 on $\mathbb{R}$. According to the remarks in the beginning of the present subsection, this will guarantee existence of blossoms in any piecewise hyperbolic tension space. Existence of B-spline bases in any piecewise hyperbolic tension spline space will then follow from Theorem 2.2.

For a given nonnegative number r, we denote by c_r, s_r the two functions in the kernel of the differential operator $D^{n-1}(D^2 - r^2 I)$ satisfying the conditions

$$c_r(0) = 1,\ c_r'(0) = 0,\ s_r(0) = 0,\ s_r'(0) = 1 \ ,$$

namely $c_r(t) = \cosh rt$, and $s_r(t) = t$ if $r = 0$, $s_r(t) = 1/r \sinh rt$ otherwise. Define w_{n-1} on the interval $[t_0, t_1]$ by $w_{n-1}(t) := c_r(t - t_0)$. For a given positive integer i, assume that we have been able to extend w_{n-1} into a strictly increasing (hence positive) C^1 function on $[t_0, t_i]$, the restriction of which to $[t_\ell, t_{\ell+1}]$ is in the kernel of the differential operator $D^2 - {r_\ell}^2 I$, for $0 \le \ell \le i-1$. Then, by setting

$$w_{n-1}(t) := w_{n-1}(t_i) c_{r_i}(t - t_i) + w_{n-1}'(t_i) s_{r_i}(t - t_i) \ ,$$

for $t \in [t_i, t_{i+1}]$, the function w_{n-1} is now clearly C^1 and strictly increasing on $[t_0, t_{i+1}]$, and its restriction to the additional interval $[t_i, t_{i+1}]$ is in the kernel of the differential operator $D^2 - {r_i}^2 I$. By induction we can therefore extend w_{n-1} on $[t_0, +\infty[$ in accordance with the three requirements: positivity, C^1, and sections in $\mathrm{Ker}(D^2 - {r_\ell}^2 I)$. Similar arguments can be used to extend w_{n-1} on $]-\infty, t_0]$, with the same three requirements. ■

§3. Existence of exponential B-spline bases: total positivity sufficient conditions

Piecewise hyperbolic tension spaces are special cases of piecewise exponential spaces as introduced in subsection 3.2 below. we shall now address this more general framework.

3.1. Exponential spaces

Given $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, we denote by $\mathcal{E}(\lambda)$ the kernel of the linear differential operator with constant coefficients $D \prod_{s=1}^n (D - \lambda_s I)$. Assuming that, up to a permutation, $\lambda = ({\ell_1}^{[r_1]}, \dots, {\ell_s}^{[r_s]})$, with pairwise distinct $\ell_1, \dots, \ell_s$, and positive integers $r_1, \dots, r_s$, it is well known that the space $D\mathcal{E}(\lambda)$ is spanned by the n functions

$$x^j\ e^{\ell_i x}, \quad 1 \le i \le s\ , \quad 0 \le j \le r_i - 1\ .$$

For this reason, we shall call $\mathcal{E}(\lambda)$ the *exponential space* associated with λ.

Let us introduce the following weight functions:

$$w_1(x) := e^{\lambda_1 x}\ , \quad w_k(x) := e^{(\lambda_k - \lambda_{k-1})x}\ , \quad k = 2, \dots, n\ . \tag{3.1}$$

If $L_1, \dots, L_{n+1}$ are the corresponding differential operators defined by means of (2.9), it can be readily checked that, for all $k = 1, \dots, n$ and all $u \in C^\infty(\mathbb{R})$,

$$L_k u(x) = e^{-\lambda_k x} \sum_{r=0}^{k-1} (-1)^r \sigma_r^{k-1}(\lambda)\ u^{(k-r)}(x)\ , \quad x \in \mathbb{R}\ , \tag{3.2}$$

where $\sigma_r^j(\lambda)$ denotes the fundamental symmetric function of degree r in the j variables $\lambda_1, \ldots, \lambda_j$, *i.e.*,

$$\sigma_r^j(\lambda) := \sum_{1\le i_1<i_2<\cdots<i_r\le j} \prod_{s=1}^{r} \lambda_{i_s} \ .$$

Accordingly, denoting by I the identity, we can write equality (3.2) as follows:

$$L_k u(x) = e^{-\lambda_k x} \prod_{s=1}^{k-1} (D-\lambda_s I)u'(x) \ , \quad x \in \mathbb{R} \ . \tag{3.3}$$

Note that the latter equality is also valid for $k = n+1$ provided that we set $\lambda_{n+1} := \lambda_n$. Therefore, we can also define the space $\mathcal{E}(\lambda)$ as $\mathcal{E}(\lambda) = \mathrm{Ker} L_{n+1}$. It follows that the space $D\mathcal{E}(\lambda)$ is an n-dimensional extended Chebyshev space on any closed bounded interval, hence on $\mathbb{R}$. The space $\mathcal{E}(\lambda)$ in turn is an $(n+1)$-dimensional extended Chebyshev space on $\mathbb{R}$.

3.2. Piecewise exponential spaces

From now on, we assume that, for each $\ell \in \mathbb{Z}$, $\mathcal{E}_\ell$ is the restriction to $[t_\ell, t_{\ell+1}]$ of an exponential space, say $\mathcal{E}(\lambda^\ell) = \mathrm{Ker}\big[D\prod_{s=1}^n (D-\lambda_s^\ell I)\big]$, where $\lambda^\ell = (\lambda_1^\ell, \ldots, \lambda_n^\ell) \in \mathbb{R}^n$. We shall refer to the corresponding spaces $\mathcal{E}$ as *piecewise exponential spaces*, and to splines based on such spaces as *piecewise exponential splines* to emphasize that the sections of such splines are chosen in different exponential spaces.

In the present situation, according to (3.3), the matrix $\mathcal{A}_\ell$ introduced in (2.10) can be written as follows:

$$\mathcal{A}_\ell(x) = \Delta_\ell(x) \ . \ \mathcal{B}(\lambda^\ell) \ , \tag{3.4}$$

where $\Delta(x)$ denotes the diagonal matrix $(e^{-\lambda_1^\ell x}, \ldots, e^{-\lambda_n^\ell x})$, and where, for any $\lambda = (\lambda_1, \ldots, \lambda_n)$

$$\mathcal{B}(\lambda) := \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -\sigma_1^1(\lambda) & 1 & 0 & \ldots & 0 \\ \sigma_2^2(\lambda) & -\sigma_1^2(\lambda) & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ (-1)^{n-1}\sigma_{n-1}^{n-1}(\lambda) & (-1)^{n-2}\sigma_{n-2}^{n-1}(\lambda) & \ldots & \ldots & 1 \end{pmatrix} . \tag{3.5}$$

On account of (2.12) and (3.4), the connection matrix $\mathcal{M}_\ell$ is given by

$$\mathcal{M}_\ell = \Delta_\ell(t_\ell) \ . \ \mathcal{B}(\lambda^\ell) \ . \ \mathcal{B}(\lambda^{\ell-1})^{-1} \ . \ \Delta_{\ell-1}(t_\ell)^{-1} \ . \tag{3.6}$$

In order to apply Theorems 2.5 and 2.6, we are interested in the total positivity of the matrices $\mathcal{M}_\ell$ and the submatrices $\widehat{\mathcal{M}}_\ell$. It is well-known that any product of totally positive matrix is totally positive. Both $\Delta_\ell(t_\ell)$ and $\Delta_{\ell-1}(t_\ell)^{-1}$ being diagonal and totally positive, the total positivity of $\mathcal{M}_\ell$ is thus equivalent to that of the product $\mathcal{B}(\lambda^\ell) \,.\, \mathcal{B}(\lambda^{\ell-1})^{-1}$. Any matrix $\widehat{\mathcal{M}}_\ell$ being built similarly to $\mathcal{M}_\ell$, we are actually in search of conditions on $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ ensuring the total positivity of a product $\mathcal{B}(\mu) \,.\, \mathcal{B}(\lambda)^{-1}$.

For any integers $0 \le i \le j \le n$, let us introduce the following quantities:

$$H_i^j(\lambda) := \sum_{\alpha_1+\cdots+\alpha_j=i} {\lambda_1}^{\alpha_1} \ldots {\lambda_j}^{\alpha_j} \ , \tag{3.7}$$

so that in particular $H_0^j(\lambda) = 1$ for all $j \le n$. Setting by convention $H_i^j(\lambda) = 0$ for $i < 0$, consider the lower triangular matrix

$$\mathcal{C}(\lambda) := \big(H_{i-j}^j(\lambda)\big)_{1\le i,j\le n} \ . \tag{3.8}$$

Proposition 3.1. *Given $\lambda = (\lambda_1, \dots, \lambda_n)$, and $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$, the coefficient of row i, column j of the lower triangular matrix*

$$\mathcal{D}(\lambda, \mu) := \mathcal{B}(\mu) \,.\, \mathcal{C}(\lambda) \ , \tag{3.9}$$

is given by

$$\mathcal{D}(\lambda, \mu)_{i,j} = [\lambda_1, \dots, \lambda_j]\vartheta_{\mu,i} \ , \quad 1 \le i, j \le n \ , \tag{3.10}$$

where the function $\vartheta_{\mu,i}$ is defined by

$$\vartheta_{\mu,i}(x) := \prod_{1 \le \ell \le i-1} (x - \mu_\ell) \ , \tag{3.11}$$

and where, for any sufficiently differentiable function f, $[\lambda_1, \dots, \lambda_j]f$ denotes the $(j-1)$th order divided difference of f based on the points $\lambda_1, \dots, \lambda_j$.

Proof: It is well-known that divided differences of products of functions can be calculated by means of the Leibniz formula

$$[\lambda_1, \dots, \lambda_j](fg) = \sum_{k=1}^{j} [\lambda_1, \dots, \lambda_k]f \ [\lambda_k, \dots, \lambda_j]g \ .$$

In the particular case $g(x) = x$, this formula reduces to

$$[\lambda_1, \dots, \lambda_j](fg) = [\lambda_1, \dots, \lambda_{j-1}]f + \lambda_j [\lambda_1, \dots, \lambda_j]f \ . \tag{3.12}$$

Setting $g_s(x) := x^s$, and using the equality $g_{s+1}(x) = x g_s(x)$, a straightforward induction based on (3.12) provides the divided differences of the functions g_s

$$[\lambda_1, \dots, \lambda_j]g_s = H^j_{s-j+1}(\lambda) \ , \quad 1 \le j \le n \ , \ s \in \mathbb{N} \ . \tag{3.13}$$

Now, from (3.9), (3.5), and (3.8), we obtain, for any $i, j = 1, \dots, n$

$$\mathcal{D}(\lambda, \mu)_{i,j} = \sum_{k=1}^{i} (-1)^{i-k} \ \sigma^{i-1}_{i-k}(\mu) \ H^j_{k-j}(\lambda). \tag{3.14}$$

On the other hand, $\vartheta_{\mu,i} = \sum_{k=1}^{i} (-1)^{i-k} \, \sigma^{i-1}_{i-k}(\mu) \, g_{k-1}$. Hence, on account of (3.13), equality (3.14) is nothing but the announced equality (3.10). ∎

Corollary 3.2. *For any $\lambda \in \mathbb{R}^n$, the matrix $\mathcal{C}(\lambda)$ is the inverse of $\mathcal{B}(\lambda)$, and therefore, for any $\mu \in \mathbb{R}^n$, the product $\mathcal{B}(\mu) \,.\, \mathcal{B}(\lambda)^{-1}$ is nothing but the matrix $\mathcal{D}(\lambda, \mu)$ given by (3.9) or (3.10).*

Proof: Given $\lambda \in \mathbb{R}^n$, according to (3.10), $\mathcal{D}(\lambda, \lambda)_{i,j} = [\lambda_1, \dots, \lambda_j]\vartheta_{\lambda,i}$.

Applying the Leibniz formula to calculate the divided differences of the product $(x - \lambda_i)\vartheta_{\lambda,i}(x) = \vartheta_{\lambda,i+1}(x)$, a simple induction shows that $\mathcal{D}(\lambda, \lambda)_{i,j} = \delta_{i,j}$. ∎

3.3. Total positivity of $\mathcal{D}(\lambda, \mu)$

In order to apply the sufficient conditions given in Theorems 2.5 and 2.6 to guarantee existence of blossoms either in piecewise exponential spaces or in piecewise exponential spline spaces, we have to prove that a certain matrix of the type $\mathcal{D}(\lambda, \mu)$ defined in (3.10) is totally positive. Given $\lambda = (\lambda_1, \dots, \lambda_n) \, \mathbb{R}^n$ and $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$, we thus want to find conditions on λ and μ which are necessary and sufficient to ensure that

$$D \begin{pmatrix} i_1, \dots, i_k \\ j_1, \dots, j_k \end{pmatrix} := \det \left([\lambda_1, \dots, \lambda_{j_s}] \vartheta_{\mu, i_r} \right)^k_{r,s=1} \ge 0,$$

for all $1 \le k \le n$, $1 \le i_1 < \dots < i_k \le n$, and $1 \le j_1 < \dots < j_k \le n$. Since $\mathcal{D}(\lambda, \mu)$ is lower triangular with ones on the diagonal many of its minors will be zero. Indeed, we have

Lemma 3.3. *Suppose $1 \le k \le n$, $1 \le i_1 < \ldots < i_k \le n$, and $1 \le j_1 < \ldots < j_k \le n$. If $j_s > i_s$ for some s, then $D = D\begin{pmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{pmatrix} = 0$.*

Proof: We have $D = \det(M)$, where $M = ([\lambda_1, \ldots, \lambda_{j_s}]\vartheta_{\mu,i_r})_{r,s=1}^k$. Since $j_s > i_s$ and ϑ_{μ,i_s} is a polynomial of degree less than i_s it follows from properties of divided differences that the (s,s)-entry $[\lambda_1, \ldots, \lambda_{j_s}]\vartheta_{\mu,i_s}$ of M is zero. This implies that the upper right s-by-$(k+1-s)$ corner of M is zero and hence det(M)=0. ■

Given functions $f_1, \ldots, f_k$ and real numbers $x_1, \ldots, x_k$ we let

$$\det\begin{pmatrix} x_1, \ldots, x_k \\ f_1, \ldots, f_k \end{pmatrix}$$

denote the determinant of the collocation matrix with element $f_i(x_j)$ in row i and column j

Lemma 3.4. *Suppose $k \in \mathbb{N}$, and $\lambda_i \ne \lambda_j$ for $i \ne j$. Then for any functions $f_1, \ldots, f_k$ defined on the λ's*

$$\det\left(([\lambda_1, \ldots, \lambda_j]f_i)_{i,j=1}^k\right) = \det\begin{pmatrix} \lambda_1, & \ldots & , \lambda_k \\ f_1, & \ldots & , f_k \end{pmatrix} / \prod_{1 \le i < j \le k} (\lambda_j - \lambda_i).$$

Proof: We use the divided difference formula

$$[\lambda_1, \ldots, \lambda_j]f = \sum_{r=1}^{j} f(\lambda_r) / \prod_{\substack{s=1 \\ s \ne r}}^{j} (\lambda_r - \lambda_s)$$

and straightforward determinant manipulations. ■

The minors of $\mathcal{D}(\lambda, \mu)$ defined from the first columns and any consecutive rows can be computed exactly.

Lemma 3.5. *For $0 \le i < i + k \le n$ we have*

$$D\begin{pmatrix} i+1, \ldots, i+k \\ 1 \quad , \ldots, \quad k \end{pmatrix} = \prod_{j=1}^{k} \vartheta_{\mu,i+1}(\lambda_j). \tag{3.15}$$

Proof: By continuity it is enough to prove (3.15) for distinct λ's. Setting $\psi_j := \vartheta_{\mu,i+j}/\vartheta_{\mu,i+1}$ for $j = 1, \ldots, k$ and using Lemma 3.4 in both directions, we obtain

$$\begin{aligned} D\begin{pmatrix} i+1, \ldots, i+k \\ 1 \quad , \ldots, \quad k \end{pmatrix} &= \det\begin{pmatrix} \lambda_1, \ldots, \lambda_k \\ \vartheta_{\mu,i+1}, \cdots, \vartheta_{\mu,i+k} \end{pmatrix} / \prod_{1 \le i < j \le k} (\lambda_j - \lambda_i) \\ &= \prod_{j=1}^{k} \vartheta_{\mu,i+1}(\lambda_j) \det\begin{pmatrix} \lambda_1, \ldots, \lambda_k \\ \psi_1, \cdots, \psi_k \end{pmatrix} / \prod_{1 \le i < j \le k} (\lambda_j - \lambda_i) \\ &= \prod_{j=1}^{k} \vartheta_{\mu,i+1}(\lambda_j) \det\left(([\lambda_1, \ldots, \lambda_j]\psi_i)_{i,j=1}^k\right). \end{aligned}$$

The last determinant is equal to one since it involves a unit lower triangular matrix. ■

From Lemma 3.5 we obtain a necessary condition for the total positivity of $\mathcal{D}(\lambda, \mu)$ in the case of distinct λ_i's and μ_i's.

Lemma 3.6. *Suppose* $\{\lambda_1,\dots,\lambda_n\}\cap\{\mu_1\dots,\mu_n\}=\emptyset$. *Then* $D\begin{pmatrix} i+1,\dots,i+k \\ 1\quad,\dots,\quad k\end{pmatrix}>0$ *for all* $0\le i<i+k\le n$ *if and only if*

$$\lambda_j-\mu_i>0,\quad \text{for all } i,j\ge 1, \text{ and } i+j\le n. \tag{3.16}$$

Proof: According to Lemma 3.5 the positivity of $D\begin{pmatrix} i+1,\dots,i+k \\ 1\quad,\dots,\quad k\end{pmatrix}>0$ for all i,k is equivalent to the positivity of the product $\prod_{j=1}^k \vartheta_{\mu,i+1}(\lambda_j)$ for $0\le i<i+k\le n$. By successively taking $k=1$ and $i=1,\dots,n-1$, $k=2$ and $i=1,\dots,n-2$, and so forth, we see that all these products are positive if and only if (3.16) holds. ■

We want to show that (3.16) is also sufficient and start with.

Lemma 3.7. *Suppose* $\{\lambda_1,\dots,\lambda_n\}\cap\{\mu_1\dots,\mu_n\}=\emptyset$. *Then* $D\begin{pmatrix} i_1,\dots,i_k \\ 1,\dots,k\end{pmatrix}>0$ *for all* $1\le k\le n$ *and all* $1\le i_1<\dots<i_k\le n$ *if and only if* (3.16) *holds.*

Proof: The fact that condition (3.6) is necessary follows from Lemma 3.6. Conversely, suppose (3.16) holds and let let us prove that $D\begin{pmatrix} i_1,\dots,i_k \\ 1,\dots,k\end{pmatrix}>0$ for given $1\le k\le n$ and $1\le i_1<\dots<i_k\le n$. Suppose there is a gap in the sequence $i_1,\dots,i_k$, *i.e.*, there are integers i,s with $i_s<i<i_{s+1}$. By [5, p.8] we have

$$\begin{aligned} &D\begin{pmatrix} i_2,\dots,i,\dots i_{k-1} \\ 1,\quad\dots,\quad k-1\end{pmatrix} D\begin{pmatrix} i_1,\dots,i_k \\ 1,\dots,k\end{pmatrix}\\ &\quad= D\begin{pmatrix} i_2,\dots,i_k \\ 1,\dots,k-1\end{pmatrix} D\begin{pmatrix} i_1,\dots,i,\dots i_{k-1} \\ 1,\quad\dots,\quad k\end{pmatrix}\\ &\qquad+ D\begin{pmatrix} i_1,\dots,i_{k-1} \\ 1,\dots,k-1\end{pmatrix} D\begin{pmatrix} i_2,\dots,i,\dots i_k \\ 1,\quad\dots,\quad k\end{pmatrix}\end{aligned}$$

By a standard argument we use induction on k and the size $g=\sum_{j=2}^k(i_j-i_{j-1}-1)$ of the gaps. All determinants except $D\begin{pmatrix} i_1,\dots,i_k \\ 1,\dots,k\end{pmatrix}$ are either of order $k-1$ or have gaps of size at most $g-1$. By Lemma 3.6 and induction all those minors are positive and it follows that $D\begin{pmatrix} i_1,\dots,i_k \\ 1,\dots,k\end{pmatrix}>0$. ■

Lemma 3.8. *Suppose* $\{\lambda_1,\dots,\lambda_n\}\cap\{\mu_1\dots,\mu_n\}=\emptyset$. *For any integers* l,k *with* $0\le l<l+k\le n$ *and any* $1\le i_1<\dots<i_k\le n$ *we consider the minor* $D\begin{pmatrix} i_1,\dots,i_k \\ l+1,\dots,l+k\end{pmatrix}$. *This determinant is equal to zero if* $i_1\le l$. *If* $i_1>l$ *then the minor is positive for all* $0\le l<l+k\le n$ *and any* $1\le i_1<\dots<i_k\le n$ *if and only if* (3.16) *holds.*

Proof: If $i_1\le l$ then the minor is zero since the row corresponding to ϑ_{μ,i_1} is zero. If $i_1>l$ then we can augment the minor as follows:

$$D\begin{pmatrix} i_1,\dots,i_k \\ l+1,\dots,l+k\end{pmatrix}=D\begin{pmatrix} 1,\dots,l,i_1,\dots,i_k \\ 1,\dots,l,l+1,\dots,l+k\end{pmatrix}.$$

This follows since the upper left l-by-l corner of the minor on the right is unit lower triangular. That the minor is positive if and only if (3.16) holds now follows from Lemma 3.7. ■

Lemma 3.9. *Suppose* $\{\lambda_1,\dots,\lambda_n\}\cap\{\mu_1\dots,\mu_n\}=\emptyset$. *Then* $D\begin{pmatrix} i_1,\dots,i_k \\ j_1,\dots,j_k\end{pmatrix}>0$ *for all* $1\le k\le n$, $1\le i_1<\dots<i_k\le n$, *and* $1\le j_1<\dots<j_k\le n$ *with* $i_s\ge j_s$ *for* $s=1,\dots,k$ *if and only if* (3.16) *holds.*

Proof: Suppose (3.16) holds and that $1 \le k \le n$, $1 \le i_1 < \cdots < i_k \le n$, and $1 \le j_1 < \cdots < j_k \le n$ with $i_s \ge j_s$ for $s = 1, \ldots, k$. We use a standard argument involving induction on k and the size $g = \sum_{i=2}^{k}(j_1 - j_2 - 1)$ of the column gaps. There are two cases. If $j_1 = i_1$ then

$$D\begin{pmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{pmatrix} = D\begin{pmatrix} i_2, \ldots, i_k \\ j_2, \ldots, j_k \end{pmatrix},$$

where the latter minor is of order $k-1$ and hence positive by induction. Suppose next $i_1 > j_1$ and $g > 0$. We pick $j \notin \{j_1, \ldots, j_k\}$ and use the basic determinant identity this time on the columns to obtain

$$\begin{aligned} &D\begin{pmatrix} i_1\ , & \ldots & , i_{k-1} \\ j_2, \ldots, & j, & \ldots, j_{k-1} \end{pmatrix} D\begin{pmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{pmatrix} \\ &= D\begin{pmatrix} i_1, \ldots, i_{k-1} \\ j_2, \ldots, j_k \end{pmatrix} D\begin{pmatrix} i_1, & \ldots & , \quad i_k \\ j_1, \ldots, & j, & \ldots j_{k-1} \end{pmatrix} \\ &+ D\begin{pmatrix} i_1, \ldots, i_{k-1} \\ j_1, \ldots, j_{k-1} \end{pmatrix} D\begin{pmatrix} i_1, & \ldots & , \quad i_k \\ j_2, \ldots, & j, & \ldots j_k \end{pmatrix}. \end{aligned}$$

Here we choose j as small as possible. This choice of j implies that $j = j_l + 1$ for some l. Now $i_s \ge i_s$ for $s \le l$ and $i_{l+1} \ge i_l + 1 \ge j_l + 1 = j$. Moreover, $i_s \ge j_s \ge j_{s-1}$ for $s = l+2, \ldots, k$. Thus the nesting assumption of the row and column indices of the first and last minor is satisfied and those minors together with $D\begin{pmatrix} i_1, \ldots, i_{k-1} \\ j_1, \ldots, j_{k-1} \end{pmatrix}$ are positive by the induction hypothesis. Since the first two minors on the right are nonnegative we conclude that $D\begin{pmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{pmatrix}$ is positive. The converse follows from Lemma 3.6 in the same way as in the proof of Lemma 3.7. ■

Theorem 3.10. *Suppose* $\{\lambda_1, \ldots, \lambda_n\} \cap \{\mu_1 \ldots, \mu_n\} = \emptyset$. *Then the matrix* $\mathcal{D}(\lambda, \mu)$ *is totally positive if and only if* (3.16) *holds.*

Proof: This follows by combining Lemma 3.3 and Lemma 3.9. ■

Consider now the case where the first λ's and μ's are equal.

Lemma 3.11. *Suppose that* $\lambda_i = \mu_i$ *for* $i = 1, \ldots, r$. *Then*

$$\mathcal{D}(\lambda, \mu) = \begin{pmatrix} I_r & 0 \\ 0 & \mathcal{D}(\widetilde{\lambda}, \widetilde{\mu}) \end{pmatrix},$$

where I_r *is the identity matrix of order* r, $\mathcal{D}(\widetilde{\lambda}, \widetilde{\mu})$ *is the matrix of order* $(n-r)$ *defined from* $\widetilde{\lambda} := (\lambda_{r+1}, \ldots, \lambda_n)$ *and* $\widetilde{\mu} := (\mu_{r+1}, \ldots, \mu_n)$ *according to* (3.10), *i.e.,*

$$\mathcal{D}(\widetilde{\lambda}, \widetilde{\mu})_{i,j} := [\lambda_{r+1}, \ldots, \lambda_{r+j}]\vartheta_{\widetilde{\mu},i}, \quad 1 \subset, j \le n - r\ , \tag{3.17}$$

with $\vartheta_{\widetilde{\mu},i}(x) := \vartheta_{\mu,r+i}(x)/\vartheta_{\mu,r+1}(x) = \prod_{j=r+1}^{r+i-1}(x - \mu_j)$.

Proof: Since $\mathcal{D}(\lambda, \mu)$ is lower triangular and $\mu_i = \lambda_i$ for $i = 1, \ldots, r$ we see that $\mathcal{D}(\lambda, \mu)$ is block diagonal. For the $\mathcal{D}(\widetilde{\lambda}, \widetilde{\mu})$ part we apply repeatedly the divided difference identity

$$[\lambda_k, \ldots, \lambda_n]\big((\cdot - \lambda_k)\, u\big) = [\lambda_{k+1}, \ldots, \lambda_n]u\ . \tag{3.18}$$

resulting from the Leibniz formula, valid for any sufficiently differentiable function u defined on $\mathbb{R}$, *i.e.*, when we take the divided difference of a function containing the factor $(x - \lambda_k)$ we can remove that factor whenever λ_k is one of the arguments of the divided difference. ■

We now have the following strengthening of Theorem 3.10.

Theorem 3.12. *Suppose for a nonnegative integer* r *that* $\lambda_i = \mu_i$ *for* $i = 1, \ldots, r$ *and that* $\{\lambda_{r+1}, \ldots, \lambda_n\} \cap \{\mu_{r+1} \ldots, \mu_n\} = \emptyset$. *Then the matrix* $D(\lambda, \mu)$ *is totally positive if and only if* $\lambda_{r+j} - \mu_{r+i} > 0$, *for all* $i, j \ge 1$ *and* $i + j \le n - r$.

Proof: From Lemma 3.11 it follows that $\mathcal{D}(\lambda, \mu)$ is totally positive if and only if the matrix $\mathcal{D}(\widetilde{\lambda}, \widetilde{\mu})$ given by (3.17) is totally positive. By Theorem 3.10, $\mathcal{D}(\widetilde{\lambda}, \widetilde{\mu})$ is totally positive if and only if $\lambda_{r+j} - \mu_{r+i} > 0$, for all $i, j \ge 1$ and $i + j \le n - r$. ■

3.4. Comments on the total positivity conditions

Theorems 3.10 and 3.12 give necessary and sufficient conditions for total positivity of certain (connection) matrices and by Theorem 2.5 the total positivity of such matrices is sufficient for the existence of $\mathcal{E}$-blossoms and hence for the existence of B-splines bases. In this subsection we shall apply these conditions to particular cases and we shall give an example showing that blossoms may exist in the space $\mathcal{E}$ even though the connection matrices are not totally positive.

Given $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and any permutation σ of the set $\{1, \dots, n\}$, let us set ${}^{\sigma}\lambda := (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)})$. Since $\mathcal{E}({}^{\sigma}\lambda) = \mathcal{E}(\lambda)$, with the set $\mathcal{E}(\lambda)$ it may possible to associate another sequence of weight funtions, defined from ${}^{\sigma}\lambda$ instead of λ according to (3.1). Given another n-tuple $\mu := (\mu_1, \dots, \mu_n)$, and another permutation σ', it may occur that the matrix $\mathcal{D}({}^{\sigma}\lambda, {}^{\sigma'}\mu)$ is totally positive while the matrix $\mathcal{D}(\lambda, \mu)$ is not. For instance, if $n = 2$ and if r and r' are two distinct nonnegative real numbers, $\mathcal{D}((r, -r), (-r', r'))$ is totally positive, $\mathcal{D}((-r, r), (r', -r'))$ is not, $\mathcal{D}((r, -r), (r', -r'))$ is totally positive if and only if $r' < r$, and $\mathcal{D}((-r, r), (-r', r'))$ is if and only if $r' > r$.

Let us denote by $\mathcal{E}(\lambda, \mu)$ the space of all C^n functions $U : \mathbb{R} \to \mathbb{R}$ for which there exist $F \in \mathcal{E}(\lambda)$, $G \in \mathcal{E}(\mu)$ such that

$$U(x) = \begin{cases} F(x) & \text{if } x \le 0 \,, \\ G(x) & \text{if } x \ge 0 \,. \end{cases}$$

The space $\mathcal{E}(\lambda, \mu)$ is a particular example of piecewise exponential space. It corresponds to selecting any subdivision $\mathcal{T} = (t_\ell)_{\ell \in \mathbb{Z}}$ of the real line, with $t_0 = 0$, and to defining $\mathcal{E}_\ell$ as the restriction of $\mathcal{E}(\lambda)$ to $[t_\ell, t_{\ell+1}]$ for any negative ℓ and the restriction of $\mathcal{E}(\mu)$ to $[t_\ell, t_{\ell+1}]$ for any nonnegative ℓ. In order to apply Theorem 2.5 to this particuliar piecewise exponential space, we only need to ensure the total positivity of the connection matrix at the point 0. Existence of blossoms in the space $\mathcal{E}(\lambda, \mu)$ will thus be ensured by the total positivity of the matrix $\mathcal{D}({}^{\sigma}\lambda, {}^{\sigma'}\mu)$, for any permutations σ, σ'. According to Theorem 3.12, this will be satisfied as soon as

$$\lambda_{\sigma(i)} = \mu_{\sigma'(i)} \text{ for } i = 1, \dots, r, \quad \{\lambda_{\sigma(r+1)}, \dots, \lambda_{\sigma(n)}\} \cap \{\mu_{\sigma'(r+1)} \dots, \mu_{\sigma'(n)}\} = \emptyset \quad (3.19)$$

$$\lambda_{\sigma(r+j)} - \mu_{\sigma'(r+i)} > 0 \text{ for all } i, j \ge 1, \; i + j \le n - r \quad (3.20)$$

When (3.19) is satisfied, the weakest among all the conditions (3.20) is obtained by choosing the two permutations as in the following Proposition.

Proposition 3.13. *Choose two permutations σ, σ' of the set $\{1, \dots, n\}$ satisfying (3.19) and also $\lambda_{\sigma(r+1)} \ge \cdots \ge \lambda_{\sigma(n)}$ and $\mu_{\sigma'(r+1)} \le \cdots \le \mu_{\sigma'(n)}$. Then, blossoms exist in the space $\mathcal{E}(\lambda, \mu)$ as soon as*

$$\lambda_{\sigma(n-i)} - \mu_{\sigma'(r+i)} > 0, \quad 1 \le i \le n - r - 1 \,. \quad (3.21)$$

Note that the conditions stated in the latter proposition are automatically satisfied if both $\mathcal{E}(\lambda)$ and $\mathcal{E}(\mu)$ are hyperbolic tensions spaces, *i.e.*, if, up to permutations, $\lambda = (0^{[n-2]}, -r, r)$ and $\mu = (0^{[n-2]}, -r', r')$, where r, r' are distinct nonnegative real numbers.

Given an integer m, $0 \le m \le n$, let us also introduce the exponential spline space $\mathcal{S}(\lambda, \mu)$ composed of all C^{n-m} functions $S : \mathbb{R} \to \mathbb{R}$ for which there exist $F \in \mathcal{E}(\lambda)$, $G \in \mathcal{E}(\mu)$ such that

$$S(x) = \begin{cases} F(x) & \text{if } x \le 0 \,, \\ G(x) & \text{if } x \ge 0 \,. \end{cases}$$

By application of Theorem 2.6, we more generally obtain the following result.

Proposition 3.14. *Blossoms exist in the spline space $\mathcal{S}(\lambda, \mu)$ as soon as*

$$\lambda_{\sigma(n-m-i)} - \mu_{\sigma'(r+i)} > 0, \quad 1 \le i \le n - m - r - 1 \,, \quad (3.22)$$

where the permutations σ, σ' are chosen as in Proposition 3.13.

We shall show in the next section that the conditions stated in Propositions 3.13 and 3.14 are not only sufficient to ensure existence of blossoms in the space $\mathcal{E}(\lambda, \mu)$ and $\mathcal{S}(\lambda, \mu)$

respectively, but also necessary. Unfortunately, if we have to consider two or more connections between different exponential spaces, we cannot take benefit of these weak conditions. Indeed, in Theorem 2.5, the weight functions are fixed once and for all in each section and we cannot change them in one section independently of the others.

Let us return to the most general piecewise exponential space $\mathcal{E}$ described at the beginning of Subsection 3.2, *i.e.*, the case where, for all $\ell \in \mathbb{Z}$, $\mathcal{E}_\ell$ is obtained by restriction of $\mathcal{E}(\lambda^\ell) = \mathrm{Ker}\big[D\prod_{s=1}^n (D-\lambda_s^\ell I)\big]$ to $[t_\ell, t_{\ell+1}]$, where $\lambda^\ell = (\lambda_1^\ell, \ldots, \lambda_n^\ell) \in \mathbb{R}^n$. We are not allowed to change the order of the λ_i^ℓ's for a given ℓ independently of the other sections. In order to be able to conclude that blossoms exist through Theorem 2.5, we can only require, by induction, the existence of a sequence σ_ℓ, $\ell \in \mathbb{Z}$, of permutations such that, for any $\ell \in \mathbb{Z}$, the matrix $\mathcal{D}({}^{\sigma_{\ell-1}}\lambda^{\ell-1}, {}^{\sigma_\ell}\lambda^\ell)$ is totally positive. However this is a very restrictive condition as we shall point out by considering the case of piecewise hyperbolic tension spaces.

Choose $n = 2$, and $\mathcal{E}(\lambda^\ell) = \mathrm{Ker}(D^3 - {r_\ell}^2 D)$, where r_ℓ, $\ell \in \mathbb{Z}$, is a sequence of nonnegative real numbers. Assume that there exists an integer $\ell \in \mathbb{Z}$ such that $r_\ell \neq r_{\ell-1}, r_{\ell+1}$, which means that we have to consider the connections at t_ℓ and $t_{\ell+1}$ at least. If we order the central parameters in decreasing order, that is, if we work with $\lambda_\ell = (r_\ell, -r_\ell)$, we have to choose $\lambda_{\ell-1} = (r_{\ell-1}, -r_{\ell-1})$ and, in order to ensure the total positivity of $\mathcal{D}(\lambda_{\ell-1}, \lambda_\ell)$ we have to require $r_\ell < r_{\ell-1}$. More generally we shall then have to require $r_{j-1} \leq r_j$ for all $j \leq \ell$. For the $(\ell+1)$th section we can choose either $\lambda_\ell = (r_{\ell+1}, -r_{\ell+1})$, with $r_{\ell+1} < r_\ell$, or $\lambda_\ell = (-r_{\ell+1}, +r_{\ell+1})$, with no condition on $r_{\ell+1}$, but this latter choice implies that the sequence r_j, $j \geq \ell+1$, has to be nondecreasing. Similar conditions appear when ordering the central parameters in increasing order, that is when starting with $\lambda_\ell = (-r_\ell, r_\ell)$. Let us summarize all the possiblities to guarantee existence of blossoms through total positivity conditions (using all possible weights defined as in (3.1))

- the whole sequence r_j, $j \in \mathbb{Z}$, is either increasing or decreasing,
- there exists an integer j_0 such that the sequence r_j, $j \leq j_0$, is decreasing, and the sequence r_j, $j \geq j_0 + 1$, is increasing.

Now, according to Proposition 2.7, we know that blossoms exist in the piecewise hyperbolic tension space $\mathcal{E}$ without any condition on the r_ℓ's. Therefore, the condition of total positivity proves to be unnecessarily strict as soon as we have to deal with two or more connections.

§4. Existence of blossoms in $\mathcal{E}(\lambda, \mu)$ and $\mathcal{S}(\lambda, \mu)$

This section is devoted to proving to prove the following result.

Theorem 4.1. *The conditions stated in Propositions* 3.13 *and* 3.14 *are necessary and sufficient to obtain existence of blossoms in the spaces $\mathcal{E}(\lambda, \mu)$ and $\mathcal{S}(\lambda, \mu)$ introduced in Subsection 3.4.*

The sufficient part was the object of Propositions 3.13 and 3.14. The necessary part will be achieved by using Theorem 2.4.

4.1. Preliminaries

The space $D\mathcal{E}(\lambda) = \mathrm{Ker}\prod_{k=1}^n (D - \lambda_k I)$ is spanned by the n functions $x^j\, e^{\ell_i x}$, $1 \leq i \leq s$, $0 \leq j \leq r_i - 1$, where, up to a permutation, $\lambda = ({\ell_1}^{[r_1]}, \ldots, {\ell_s}^{[r_s]})$, with pairwise distinct $\ell_1, \ldots, \ell_s$, and positive integers $r_1, \ldots, r_s$. For a given $x \in \mathbb{R}$, denote by f_x the function $f_x(t) := e^{x\,t}$, $t \in \mathbb{R}$. Since the divided difference $[\lambda_1, \ldots, \lambda_n]f_x$ is a linear combination of the quantities $f_x(\ell_q), \ldots, {f_x}^{(r_q-1)}(\ell_q)$, with nonzero coefficients, as a basis of $D\mathcal{E}(\lambda)$ we can as well choose the n functions $[\lambda_1]f_{\boldsymbol{.}}\,,\ [\lambda_1, \lambda_2]f_{\boldsymbol{.}}\,, \ldots,\ [\lambda_1, \ldots, \lambda_n]f_{\boldsymbol{.}}$.

In this section, for any positive integer i, the function $x \mapsto x^i$ will be denoted $\boldsymbol{.}^i$.

Lemma 4.2. *Denote by $\Phi_\lambda : \mathbb{R} \to \mathbb{R}^n$ a given (nondegenerate) $\mathcal{E}(\lambda)$-function such that*

$$\Phi_\lambda{}'(x) = ([\lambda_1]f_x, [\lambda_1, \lambda_2]f_x, \ldots, [\lambda_1, \ldots, \lambda_n]f_x)^T\ , \quad x \in \mathbb{R}\ . \tag{4.1}$$

Then the function $\Phi_\lambda^\sharp$ *(defined according to* (2.6)*) is given by*

$$\Phi_\lambda{}^\sharp(x) = \big([\lambda_1,\ldots,\lambda_n]f_{-x}, [\lambda_2,\ldots,\lambda_n]f_{-x},\ldots,[\lambda_n]f_{-x}\big)^T\ , \quad x\in\mathbb{R}\ , \tag{4.2}$$

$$= \big([\lambda_1,\ldots,\lambda_n](f_{-x}\,\vartheta_{\lambda,1}),\ldots,[\lambda_1,\ldots,\lambda_n](f_{-x}\,\vartheta_{\lambda,n})\big)^T\ , \tag{4.3}$$

where the functions $\vartheta_{\lambda,i}$ *are defined as in* (3.11)*, i.e.,* $\vartheta_{\lambda,s}(t) := \prod_{\ell=1}^{s-1}(t-\lambda_\ell)$.

Proof: From (4.1) we obtain, for any positive integer i

$$\Phi_\lambda^{(i)}(x) = \big([\lambda_1](\mathbf{.}^{i-1}f_x)\,,\,[\lambda_1,\lambda_2](\mathbf{.}^{i-1}f_x)\,,\,\ldots\,,\,[\lambda_1,\ldots,\lambda_n](\mathbf{.}^{i-1}f_x)\,\big)^T\ ,\quad x\in\mathbb{R}\ .$$

Hence, for a given $x\in\mathbb{R}$, $\Phi_\lambda^\sharp(x) := \big(\Phi_{\lambda,1}{}^\sharp(x),\ldots,\Phi_{\lambda,n}{}^\sharp(x)\big)^T$ is uniquely determined by the n relations

$$\sum_{k=1}^n[\lambda_1,\ldots,\lambda_k](\mathbf{.}^{i-1}f_x)\ \Phi_{\lambda,k}{}^\sharp(x) = \begin{cases}0 & \text{if } 1\le i\le n-1\ ,\\ 1 & \text{if } i=n\ .\end{cases}$$

Now, using the Leibniz formula, we have

$$\sum_{k=1}^n[\lambda_1,\ldots,\lambda_k](\mathbf{.}^{i-1}f_x)\ [\lambda_k,\ldots,\lambda_n]f_{-x} = [\lambda_1,\ldots,\lambda_n](\mathbf{.}^{i-1}f_xf_{-x}) = [\lambda_1,\ldots,\lambda_n](\mathbf{.}^{i-1})\ .$$

The right-hand side of the latter equality is equal to 0 if $1\le i\le n-1$, and to 1 if $i=n$. Therefore $\Phi_{\lambda,k}{}^\sharp(x) = [\lambda_k,\ldots,\lambda_n]f_{-x}$ for all $k=1,\ldots,n$.

On the other hand, applying repeatedly (3.18) transforms (4.2) into (4.3). ∎

Lemma 4.3. *With the same notations as in Lemma 4.2, let* Φ *be any* $\mathcal{E}(\lambda,\mu)$*-function such that* $\Phi'(x) = \Phi_\lambda{}'(x)$ *for all* $x\le 0$*. Then, for all* $x\ge 0$*, we have*

$$\Phi'(x) = \mathcal{D}(\mu,\lambda)^T\ .\ \Phi_\mu{}'(x)\ , \tag{4.4}$$

where $\mathcal{D}(\mu,\lambda)$ *is the* n*th order square matrix defined according to* (3.9) *and where* Φ_μ *denotes an* $\mathcal{E}(\mu)$*-function such that* $\Phi_\mu{}'(x) := ([\mu_1]f_x, [\mu_1,\mu_2]f_x,\ldots,[\mu_1,\ldots,\mu_n]f_x)^T$.

Proof: Since the components of $\Phi_\mu{}'$ form a basis of $D\mathcal{E}(\mu)$, we know that

$$\Phi'(x) = \mathcal{A}\ .\ \Phi_\mu{}'(x),\quad x\ge 0\ , \tag{4.5}$$

where the nth order square matrix $\mathcal{A}$ is to be determined so as to ensure

$$\Phi^{(i)}(0^+) = \Phi^{(i)}(0^-)\ ,\quad i=1,\ldots,n\ . \tag{4.6}$$

Since $\Phi'(x) = \Phi_\lambda{}'(x)$ for $x\le 0$, equality (4.1) yields

$$\Phi^{(j)}(0^-) = \big([\lambda_1](\mathbf{.}^{j-1})\,,\,[\lambda_1,\lambda_2](\mathbf{.}^{j-1})\,,\,\ldots\,,\,[\lambda_1,\ldots,\lambda_n](\mathbf{.}^{j-1})\,\big)^T\ ,\quad j\ge 1\ .$$

Hence, we have $\big(\Phi'(0^-),\ldots,\Phi^{(n)}(0^-)\big) = \big([\lambda_1,\ldots,\lambda_i](\mathbf{.}^{j-1})\big)_{i,j=1,\ldots,n}$, *i.e.*, according to (3.13) and (3.8)

$$\big(\Phi'(0^-),\ldots,\Phi^{(n)}(0^-)\big) = \big(H^i_{j-i}(\lambda)\big)_{i,j=1,\ldots,n} = \mathcal{C}(\lambda)^T\ . \tag{4.7}$$

From (4.5) we can derive similarly

$$\big(\Phi'(0^+),\ldots,\Phi^{(n)}(0^+)\big) = \mathcal{A}\ .\ \mathcal{C}(\mu)^T\ . \tag{4.8}$$

Let us recall that the inverse of matrix $\mathcal{C}(\mu)$ is $\mathcal{B}(\mu)$ (cf. Corollary 3.2). Therefore, taking into account (4.7) and (4.8), relations (4.6) are fulfilled if and only if

$$\mathcal{A} = \mathcal{C}(\lambda)^T\ .\ \mathcal{C}(\mu)^{-T} = \big(\mathcal{B}(\mu)\ .\ \mathcal{C}(\lambda)\big)^T = \mathcal{D}(\mu,\lambda)^T\ .\quad ∎$$

Lemma 4.4. *Consider again an $\mathcal{E}(\lambda,\mu)$-function Φ such that $\Phi'(x) = {\Phi_\lambda}'(x)$ for all $x \leq 0$. Then, on $[0,+\infty[$, the function $\Phi^\sharp$ is given by*

$$\Phi^\sharp(x) = \big([\mu_1,\dots,\mu_n](f_{-x}\,\vartheta_{\lambda,1})\,,\ \dots\ ,\ [\mu_1,\dots,\mu_n](f_{-x}\,\vartheta_{\lambda,n})\big)^T\ . \tag{4.9}$$

Proof: For a given $x \geq 0$, $\Phi^\sharp(x) := \big(\Phi_1^\sharp(x),\dots,\Phi_n^\sharp(x)\big)^T$ is determined by the n relations $\langle\, \Phi^\sharp(x), \Phi^{(k)}(x)\,\rangle = \delta_{n,k}$, $k = 1,\dots,n$. Now, from (4.4) we can derive

$$\Phi^{(k)}(x) = \mathcal{D}(\mu,\lambda)^T\ .\ {\Phi_\mu}^{(k)}(x)\ ,\quad k = 1,\dots,n\ .$$

One can readily check that this implies

$$\Phi^\sharp(x) = \mathcal{D}(\mu,\lambda)^{-1}\ .\ {\Phi_\mu}^\sharp(x)\ . \tag{4.10}$$

Since $\mathcal{D}(\mu,\lambda) = \mathcal{B}(\mu)\,.\,\mathcal{C}(\lambda)$, it results from Corollary 3.2 that $\mathcal{D}(\mu,\lambda)^{-1} = \mathcal{D}(\lambda,\mu)$. On the other hand, from Lemma 4.2 we know that

$${\Phi_\mu}^\sharp(x) = \big([\mu_1,\dots,\mu_n]f_{-x}, [\mu_2,\dots,\mu_n]f_{-x},\dots,[\mu_n]f_{-x}\big)^T\ ,\quad x \in \mathbb{R}\ .$$

Therefore, using Proposition 3.1 and (4.2), we can write equality (4.10) as follows:

$$\Phi_i^\sharp(x) = \sum_{k=1}^n [\mu_1,\dots,\mu_k]\vartheta_{\lambda,i}\ [\mu_k,\dots,\mu_n]f_{-x}\ ,\quad i = 1,\dots,n\ .$$

Due to the Leibniz formula, the right-hand side of the latter equality is just $[\mu_1,\dots,\mu_n](f_{-x}\,\vartheta_{\lambda,i})$, which gives the announced result. ■

4.2. Existence of blossoms in the space $\mathcal{E}(\lambda,\mu)$

If blossoms exist in the space $\mathcal{E}(\lambda,\mu)$, for any $x < 0$, any $y > 0$ and any integers i, j, k such that $i + j + k = n$, the osculating flats $\mathrm{Osc}_{n-i}\Phi(x)$, $\mathrm{Osc}_{n-j}\Phi(0)$, and $\mathrm{Osc}_{n-k}\Phi(y)$ intersect at a single point. This subsection gives necessary condition for this to be satisfied.

Proposition 4.5. *Assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, and that $\{\lambda_1,\dots,\lambda_n\} \cap \{\mu_1,\dots,\mu_n\} = \emptyset$. Let Φ denote a nondegenerate $\mathcal{E}(\lambda,\mu)$-function, and let i, j, k be three nonnegative integers such that $i + j + k = n$. Suppose that, for every $x < 0$ and every $y > 0$, the intersection*

$$\mathrm{Osc}_{n-i}\Phi(x) \cap \mathrm{Osc}_{n-j}\Phi(0) \cap \mathrm{Osc}_{n-k}\Phi(y) \tag{4.11}$$

consists of a single point. Then we have

$$K_{i,k} := \prod_{r=1}^{k}\ \prod_{s=1}^{i}(\lambda_s - \mu_r) > 0\ . \tag{4.12}$$

Proof: Without any loss of generality, we can assume that $\Phi'(x) = {\Phi_\lambda}'(x)$ for $x \leq 0$. According to Theorem 2.4, given $x < 0$ and $y > 0$, the intersection (4.11) consists of a single point if and only if the n vectors $\Phi^\sharp(x),\dots,\Phi^{\sharp\,(i-1)}(x), \Phi^\sharp(0),\dots,\Phi^{\sharp\,(j-1)}(0), \Phi^\sharp(y),\dots,\Phi^{\sharp\,(k-1)}(y)$, are linearly independent, or equivalently, if and only if the determinant

$$\Delta(x,y) := \det\big(\Phi^\sharp(x),\dots,\Phi^{\sharp\,(i-1)}(x), \Phi^\sharp(0), \Phi^{\sharp\,\prime}(0^-),\dots,\Phi^{\sharp\,(j-1)}(0^-), \Phi^\sharp(y),\dots,\Phi^{\sharp\,(k-1)}(y)\big)$$

never vanishes, and therefore keeps a strict sign, on $]-\infty,0[\times]0,+\infty[$.

From (4.3) we see that for any $x < 0$, and any $r \geq 0$

$$(-1)^r\,\Phi^{\sharp\,(r)}(x) = \big([\lambda_1,...,\lambda_n](\centerdot^r f_{-x}\,\vartheta_{\lambda,1}), [\lambda_1,...,\lambda_n](\centerdot^r f_{-x}\,\vartheta_{\lambda,2}),\dots,[\lambda_1,...,\lambda_n](\centerdot^r f_{-x}\,\vartheta_{\lambda,n})\big)^T\ .$$

We also have

$$\Phi^\sharp(0) = \big([\lambda_1,\ldots,\lambda_n](1\!\!1),[\lambda_2,\ldots,\lambda_n](1\!\!1),\ldots,[\lambda_n](1\!\!1)\big)^T = (0,\ldots,0,1)^T \ ,$$

and, for $0 < r \le n-1$,

$$\begin{aligned}(-1)^r\,\Phi^{\sharp\,(r)}(0^-) &= \big([\lambda_1,\ldots,\lambda_n](\cdot^{\,r})\,,\,[\lambda_2,\ldots,\lambda_n](\cdot^{\,r})\,,\,\ldots\,,\,[\lambda_n](\cdot^{\,r})\big)^T\\ &= (\underbrace{0,\ldots,0}_{n-r-1 \text{ times}},1,\times,\ldots,\times)^T \ ,\end{aligned}$$

where $\times,\ldots,\times$ denotes quantities the values of which are not important in the present context. On account of the latter equalities, it follows that, up to a possible change of sign, we can express $\Delta(x,y)$ as a determinant of order $i+k$, namely

$$\Delta(x,y) = \det\big(C_1(x),\ldots,C_i(x),D_1(y),\ldots,D_k(y)\big) \ , \quad (x,y) \in]-\infty,0[\times]0,+\infty[\ , \tag{4.13}$$

where, for all $r \ge 1$

$$C_r(x) := [\lambda_1,\ldots,\lambda_n](\cdot^{\,r-1}\,f_{-x}\ \Theta) \ , \quad D_r(y) := [\mu_1,\ldots,\mu_n](\cdot^{\,r-1}\,f_{-y}\ \Theta) \ , \tag{4.14}$$

and the function $\Theta : \mathbb{R} \to \mathbb{R}^{i+k}$ being defined by

$$\Theta(t) := \big(\vartheta_{\lambda,1}(t)\,,\,\ldots\,,\,\vartheta_{\lambda,i+k}\,(t)\big)^T \ , \quad t \in \mathbb{R}\,. \tag{4.15}$$

We thus have to prove that if the determinant $\Delta(x,y)$ given in (4.13) keeps a constant strict sign, on $]-\infty,0[\times]0,+\infty[$, then the number $K_{i,k}$ is positivity. For the sake of simplicity, we shall first give a detailed proof in the case where $\lambda_1,\ldots,\lambda_n$ are all distinct, and so are $\mu_1,\ldots,\mu_n$.

-1) Assume that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, $\mu_1 < \mu_2 < \cdots < \mu_n$.

Then, we can write the second equality in (4.14) as follows:

$$D_r(y) = \sum_{\ell=1}^{n} e^{-\mu_\ell y}\ {\mu_\ell}^{r-1}\ Y_\ell \ , \quad Y_\ell := \frac{1}{\displaystyle\prod_{\substack{1\le s\le n\\ s\ne\ell}} (\mu_\ell-\mu_s)}\ \Theta(\mu_\ell) \ . \tag{4.16}$$

By linearity, equality (4.13) thus becomes

$$\Delta(x,y) = \sum_{1\le\ell_1,\ldots,\ell_k\le n} e^{-(\mu_{\ell_1}+\mu_{\ell_2}+\cdots+\mu_{\ell_k})y}\ \mu_{\ell_2}{\mu_{\ell_3}}^2\ldots{\mu_{\ell_k}}^{k-1}\ \det\big(C_1(x),\ldots,C_i(x),Y_{\ell_1},\ldots,Y_{\ell_k}\big) \ . \tag{4.17}$$

Since $\mu_1 < \cdots < \mu_n$, it follows that

$$\begin{aligned}\gamma(x) :&= \lim_{y\to+\infty} e^{(\mu_1+\mu_2+\cdots+\mu_k)y}\ \Delta(x,y) &(4.18)\\ &= \sum_{\{\ell_1,\ldots,\ell_k\}=\{1,\ldots,k\}} \mu_{\ell_2}{\mu_{\ell_3}}^2\ldots{\mu_{\ell_k}}^{k-1}\ \det\big(C_1(x),\ldots,C_i(x),Y_{\ell_1},\ldots,Y_{\ell_k}\big)\\ &= \det\Big(C_1(x),\ldots,C_i(x),\sum_{\ell=1}^k Y_\ell,\sum_{\ell=1}^k \mu_\ell\ Y_\ell\ldots,\sum_{\ell=1}^k {\mu_\ell}^{k-1}\ Y_\ell\Big) \ . &(4.19)\end{aligned}$$

We can write the matrix involved in (4.19) as

$$\big(C_1(x),\ldots,C_i(x),Y_1,\ldots,Y_k\big)\ .\ \begin{pmatrix}\mathcal{I}_i & 0\\ 0 & \mathcal{B}\end{pmatrix} \ ,$$

$\mathcal{B}$ being the kth order Vandermonde matrix based on $\mu_1, \ldots, \mu_k$, and $\mathcal{I}_i$ the identity matrix of order i. Hence, from (4.19) and from the second equality in (4.16) we can deduce that

$$\gamma(x) = \frac{\displaystyle\prod_{1 \le r < s \le k} (\mu_s - \mu_r)}{\displaystyle\prod_{1 \le \ell \le k} \prod_{\substack{1 \le s \le n \\ s \ne \ell}} (\mu_\ell - \mu_s)} \; \delta(x) \; , \tag{4.20}$$

with

$$\delta(x) := \det\bigl(C_1(x), \ldots, C_i(x)), \Theta(\mu_1), \ldots, \Theta(\mu_k)\bigr) \; , \quad x < 0 \; . \tag{4.21}$$

1a) <u>Let us first examine how $\delta(x)$ behaves when $x \to -\infty$</u>. Writing

$$C_r(x) = \sum_{\ell=1}^{n} e^{-\lambda_\ell x} \, {\lambda_\ell}^{r-1} \, X_\ell \; , \quad X_\ell := \frac{1}{\displaystyle\prod_{\substack{1 \le s \le n \\ s \ne \ell}} (\lambda_\ell - \lambda_s)} \; \Theta(\lambda_\ell) \; , \tag{4.22}$$

a similar argument will enable us to derive from (4.21)

$$A := \lim_{x \to -\infty} e^{(\lambda_1 + \cdots + \lambda_i)x} \; \delta(x) \tag{4.23}$$

$$= \sum_{\{\ell_1, \ldots, \ell_i\} = \{1, \ldots, i\}} \lambda_{\ell_2} {\lambda_{\ell_3}}^2 \ldots \lambda_{\ell_i}^{i-1} \; \det\bigl(X_{\ell_1}, \ldots, X_{\ell_i}, \Theta(\mu_1), \ldots, \Theta(\mu_k)\bigr)$$

$$= \frac{\displaystyle\prod_{1 \le r < s \le i} (\lambda_s - \lambda_r)}{\displaystyle\prod_{1 \le \ell \le i} \prod_{\substack{1 \le s \le n \\ s \ne \ell}} (\lambda_\ell - \lambda_s)} \; B \; , \tag{4.24}$$

where

$$B := \det\bigl(\Theta(\lambda_1), \ldots, \Theta(\lambda_i), \Theta(\mu_1), \ldots, \Theta(\mu_k)\bigr) \; . \tag{4.25}$$

$$= (-1)^{ik} \; K_{i,k} \prod_{1 \le r < s \le i} (\lambda_s - \lambda_r) \prod_{1 \le r < s \le k} (\mu_s - \mu_r) \; . \tag{4.26}$$

The latter equality comes from the fact that, for any $x_1, \ldots, x_{i+k}$

$$\det\bigl(\Theta(x_1), \ldots, \Theta(x_{i+k})\bigr) = \prod_{1 \le r < s \le i+k} (x_s - x_r) \; , \tag{4.27}$$

which is easy to check by induction on the order of the determinant. Due to our assumptions on the λ_s' and the μ_s's, equalities (4.24) and (4.26) prove that the number A is not equal to 0 and that its sign is given by

$$\operatorname{sign}(A) = (-1)^{ik} \; (-1)^{i(i-1)/2} \; \operatorname{sign}(K_{i,k}) \; . \tag{4.28}$$

1b) <u>Let us now examine what occurs when $x \to 0^-$</u>. We start again with equality (4.21). Using the Taylor series expansion of e^{-xt}, it is easy to check that each column $C_r(x)$ can be written as follows:

$$C_r(x) = \sum_{\ell=0}^{+\infty} \frac{(-x)^\ell}{\ell!} \; C_{\ell+r}(0) \; , \quad r = 1, \ldots, i \; .$$

This yields

$$\delta(x) = \sum_{\ell_1, \ldots, \ell_i \ge 0} \frac{(-x)^{\ell_1 + \cdots + \ell_i}}{\ell_1! \ldots \ell_i!} \; \det\bigl(C_{\ell_1+1}(0), \ldots, C_{\ell_i+i}(0), \Theta(\mu_1), \ldots, \Theta(\mu_k)\bigr) \; . \tag{4.29}$$

Now, due to (4.14),

$$C_r(0) = 0 \quad \text{for } 1 \leq r \leq j \ , \quad C_r(0) = \big(\underbrace{0, \ldots, 0}_{n-r \text{ times}}, 1, \times, \ldots, \times \big)^T \quad \text{for } j+1 \leq r \leq n \ .$$

Accordingly, in the series appearing in (4.29), the lowest possible degree terms are obtained for $\{\ell_1 + 1, \ldots, \ell_i + i\} = \{j+1, \ldots, j+i\}$. Let us set $\widetilde{\Theta}(t) := \big(\vartheta_{\lambda,1}(t), \ldots, \vartheta_{\lambda,k}(t)\big)^T$. Then, equality (4.29) gives

$$\delta(x) = (-x)^{ij} \ (-1)^{ik} \ (-1)^{i(i-1)/2} \ K \ \det\big(\widetilde{\Theta}(\mu_1), \ldots, \widetilde{\Theta}(\mu_k)\big) \ + \ x^{ij} \alpha(x) \ , \tag{4.30}$$

where $\alpha(x)$ goes to 0 as $x \to 0^-$, and with

$$K := \sum_{\sigma} \frac{\varepsilon(\sigma)}{(j + \sigma(1) - 1)! \ldots (j + \sigma(i) - i)!} \ , \tag{4.31}$$

the sum being taken over all permutations σ of $\{1, \ldots, i\}$ such that $j + \sigma(\ell) - \ell \geq 0$, $\ell = 1, \ldots, i$. Using relation (4.27), we know that

$$\det\big(\widetilde{\Theta}(\mu_1), \ldots, \widetilde{\Theta}(\mu_k)\big) = \prod_{1 \leq r < s \leq k} (\mu_s - \mu_r) \ . \tag{4.32}$$

Furthermore, according to Lemma 4.6 below, the number K is positive. Hence, (4.30) shows that

$$(-1)^{ik} \ (-1)^{i(i-1)/2} \ \delta(x) > 0 \ , \quad x \text{ close to } 0^-. \tag{4.33}$$

Taking equalities (4.18), (4.20), and (4.23) into consideration, from $\Delta(x,y)$ keeping the same strict sign on $]-\infty, 0[\times]0, +\infty[$ and from (4.33) we can conclude that

$$(-1)^{ik} \ (-1)^{i(i-1)/2} \ A > 0 \ , \tag{4.34}$$

Thanks to (4.28), we eventually have proved the positivity of $K_{i,k}$.

2) Suppose now that we have only $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$.

We can thus write λ and μ as follows:

$$\lambda = (a_1{}^{[r_1]}, \ldots, a_{p_1}{}^{[r_{p_1}]}) \ , \quad \mu = (b_1{}^{[s_1]}, \ldots, b_{q_1}{}^{[s_{q_1}]}) \ ,$$

with positive exponents $r_1, \ldots, r_{p_1}$, $s_1, \ldots, s_{q_1}$, and $a_1 > \cdots > a_{p_1}$, $b_1 > \cdots > b_{q_1}$. We can assume that $i, k \geq 1$ (otherwise there is nothing to prove). Hence, we also have

$$\lambda_i = (a_1{}^{[r_1]}, \ldots, a_p{}^{[r_p]}, a_{p+1}{}^{[\alpha]}) \ , \quad \mu_k = (b_1{}^{[s_1]}, \ldots, b_q{}^{[s_q]}, b_{q+1}{}^{[\beta]}) \ ,$$

for some $p < p_1$, $q < q_1$, and with $0 \leq \alpha < r_{p+1}$, $0 \leq \beta < s_{q+1}$. The method will be similar to our case 1), but we shall now break the symmetry between $\lambda_1, \ldots, \lambda_n$ and between $\mu_1, \ldots, \mu_n$ in the expression of the divided differences involved in $\Delta(x,y)$. Note that this expression may now contain exponentials multiplied by powers of x, y. After multiplication of $\Delta(x,y)$ by $e^{(\mu_1 + \mu_2 + \cdots + \mu_k)y}$ and possibly division by a suitable power of y, if we let y go to $+\infty$, up to a multiplicative constant, we shall now have to deal with the function δ defined on $]-\infty, 0[$ by

$$\begin{aligned} \delta(x) := \det\big(&C_1(x), \ldots, C_i(x), \\ &\Theta(b_1), \ldots, \Theta^{(s_1-1)}(b_1), \ldots, \Theta^{(s_q-1)}(b_q), \Theta(b_{q+1}), \ldots, \Theta^{(\beta-1)}(b_q)\big) \ . \end{aligned} \tag{4.35}$$

We do not give details on this part of the proof, for the argument is similar to what we shall do now with $x \to -\infty$ instead of $y \to +\infty$.

2a) Let first x go to $-\infty$. Making a repeated use of the equality

$$[\lambda_\ell, \dots, \lambda_n](\cdot^{\,r} f_{-x}\Theta) = \lambda_\ell \; [\lambda_\ell, \dots, \lambda_n](\cdot^{\,r-1} f_{-x}\Theta) + [\lambda_{\ell+1}, \dots, \lambda_n](\cdot^{\,r-1} f_{-x}\Theta) \ ,$$

leads to the following expression of $\delta(x)$:

$$\begin{aligned}\delta(x) = (-1)^{i(i-1)/2} \ \det\big(&[\lambda_i, \dots, \lambda_n](f_{-x}\Theta), [\lambda_{i-1}, \dots, \lambda_n](f_{-x}\Theta), \dots, \\ &[\lambda_2, \dots, \lambda_n](f_{-x}\Theta)), [\lambda_1, \dots, \lambda_n](f_{-x}\Theta)), \Theta(b_1), \dots, \Theta^{(\beta-1)}(b_{q+1})\big) \ . \quad (4.36)\end{aligned}$$

Before letting x go to $-\infty$ we have to identify the elements which contains $e^{-(\lambda_1+\cdots+\lambda_i)x} = e^{-(r_1 a_1+\cdots+r_p a_p+\alpha a_{p+1})x}$ and the greatest possible powers of x.

Suppose first that $i \geq r_1$ (*i.e.*, $p' \geq 1$). Then in order to produce terms containing $e^{-(\lambda_1+\cdots+\lambda_i)x}$, we necessarily have to involve the r_1 columns $[\lambda_\ell, \dots, \lambda_n](f_{-x}\Theta))$, $1 \leq \ell \leq r_1$, in (4.36). Now, for $1 \leq \ell \leq r_1$, we can write

$$\begin{aligned}[\lambda_{r_1-\ell+1}, \dots, \lambda_n](f_{-x}\Theta) &= [a_1{}^{[\ell]}, a_2{}^{[r_2]}, \dots, a_p{}^{[r_p]}](f_{-x}\Theta) \\ &= e^{-a_1 x} \sum_{r=0}^{\ell-1} A_{\ell,r} \sum_{s=0}^{r} (-x)^{r-s} \tbinom{r}{s} \ \Theta^{(s)}(a_1) + \cdots\cdots , \quad (4.37)\end{aligned}$$

with

$$A_{\ell,\ell-1} = \frac{1}{(\ell-1)! \displaystyle\prod_{s=2}^{p_1} (a_1 - a_s)^{r_s}} \ , \quad 1 \leq \ell \leq r_1 \ . \quad (4.38)$$

and the dots in (4.37) standing for terms which do not contain $e^{-a_1 x}$. Therefore, in the search of all terms of $\delta(x)$ containing $e^{-(\lambda_1+\cdots+\lambda_i)x}$, in (4.36) we can replace the columns $[\lambda_\ell, \dots, \lambda_n](f_{-x}\Theta))$, $1 \leq \ell \leq r_1$, by

$$e^{-r_1 a_1 x} \ A_{1,0} A_{2,1} \dots A_{r_1,r_1-1} \ \det\big(\dots, \Theta(a_1), \dots, \Theta^{(r_1-1)}(a_1), \Theta(b_1), \dots\big) \ . \quad (4.39)$$

On the opposite, if $i < r_1$ (*i.e.*, if $p = 0$ and $i = \alpha$), we obtain

$$\delta(x) = e^{-\alpha a_1 x} \Bigg((-x)^{\alpha(r_1-\alpha)} \ \Lambda_{\alpha, r_1-\alpha} \prod_{\ell=r_1-\alpha+1}^{r_1} A_{\ell,\ell-1} \ \det(\Theta(a_1), \dots, \Theta^{(\alpha-1)}(a_1)) + \cdots \Bigg)$$

the dots standing for terms which contain powers of $(-x)$ of smaller exponents, and $\Lambda_{s,r} := \det\big(\binom{r+\ell}{m}\big)_{0 \leq \ell, m \leq s-1}$. It is easy to check (by induction) that $\Lambda_{s,r} = 1$. Making repeated use of such arguments, in any case we eventually obtain

$$(-1)^{i(i-1)/2} \ \delta(x) = \Gamma \ e^{-(\lambda_1+\cdots+\lambda_i)x} \ (-x)^{\alpha(r_{p'+1}-\alpha)} \ B + \cdots \ , \quad (4.40)$$

where the dots stand for terms of the form $e^{-\sigma x} x^s$, with either $\sigma < \lambda_1+\cdots+\lambda_i$ or $\sigma = \lambda_1+\cdots+\lambda_i$ and $s < \alpha(r_{p'+1} - \alpha)$, and where the constants Γ, B are defined respectively by

$$\Gamma := (-1)^{\sum_{1\leq \ell \leq s} r_\ell r_s} (-1)^{\alpha(i-\alpha)} \prod_{s=1}^{p} \prod_{\ell=1}^{r_s} A^s_{\ell,\ell-1} \prod_{\ell=r_{p+1}-\alpha+1}^{r_{p+1}} A^{p+1}_{\ell,\ell-1} \ , \quad (4.41)$$

and

$$B := \det\big(\Theta(a_1), \dots, \Theta^{(r_1-1)}(a_1), \dots, \Theta^{(r_p-1)}(a_p), \Theta(a_{p+1}), \dots, \Theta^{(\alpha-1)}(a_{p+1}), \Theta(b_1), \dots\big) \quad (4.42)$$

Therefore

$$A := \lim_{x \to -\infty} \frac{e^{(\lambda_1+\cdots+\lambda_i)x}}{(-x)^{\alpha(r_{p'+1}-\alpha)}} \delta(x) = (-1)^{i(i-1)/2} \ \Gamma \ B \ . \quad (4.43)$$

Given a function of ν variables, supposed to be antisymmetric and infinitely many times differentiable on $\mathbb{R}^\nu$, pairwise distinct numbers $\xi_1, \ldots, \xi_r$, and positive integers $\varrho_1, \ldots, \varrho_r$ such that $\sum_{s=1}^r \varrho_s = \nu$, we have (see the proof of [11, proof of Lemma 3.1])

$$\lim_{\substack{(x_1,\ldots,x_\nu)\to(\xi_1{}^{[\varrho_1]},\ldots,\xi_r{}^{[\varrho_r]})\\ x_r\neq x_s}} \frac{F(x_1,\ldots,x_\nu)}{\prod\limits_{1\le \ell<s\le\nu}(x_s-x_\ell)} = \frac{1}{\prod\limits_{\ell=1}^{r} 1!2!\ldots(\varrho_\ell-1)!}\;\frac{1}{\prod\limits_{1\le\ell<s\le r}(\xi_s-\xi_\ell)^{\varrho_\ell\varrho_s}}$$

$$\times\, \partial_{1^0,2^1,\ldots,\varrho_1{}^{\varrho_1-1},(\varrho_1+1)^0,(\varrho_1+2)^1,\ldots,n^{\varrho_r-1}} H(\xi_1{}^{[\varrho_1]},\ldots,\xi_r{}^{[\varrho_r]})\ . \tag{4.44}$$

Let us apply equality (4.44) to $F(x_1, \ldots, x_{i+k}) := \det\big(\Theta(x_1), \ldots, \Theta(x_{i+k})\big)$. On account of (4.27) and of the order of $a_1, \ldots, a_{p+1}$ and $b_1, \ldots, b_{q+1}$, there exists a positive number H such that

$$1 = (-1)^{\sum_{1\le\ell\le s} r_\ell r_s}(-1)^{\alpha(i-\alpha)} H\, \frac{B}{(-1)^{ik}\, K_{i,k}}\ , \tag{4.45}$$

From the positivity of all numbers $A^s_{\ell,\ell-1}$ and from (4.41), (4.43), and (4.45), we can conclude that the number A is not equal to 0 and that (4.28) is still valid.

2b) <u>Let us examine what occurs when $x \to 0^-$.</u> Following the same arguments as in 1b) will lead us to

$$\delta(x) = (-x)^{ij}\ (-1)^{ik}\ (-1)^{i(i-1)/2}\ K \det\big(\widetilde\Theta(b_1), \ldots, \widetilde\Theta^{(\beta-1)}(b_{q'+1})\big)\ +\ x^{ij}\alpha(x)\ , \tag{4.46}$$

where $\alpha(x)$ goes to 0 as $x \to 0^-$ and where K is the the positive number introduced in (4.31). Applying (4.44) to the function $\widetilde F(x_1, \ldots, x_k) := \det\big(\widetilde\Theta(x_1), \ldots, \widetilde\Theta(x_k)\big)$, and taking (4.32) into account, proves the positivity of $\det\big(\widetilde\Theta(b_1), \ldots, \widetilde\Theta^{(\beta-1)}(b_{q'+1})\big)$. Comparison of (4.28) and (4.46) allows to conclude to the positivity of $K_{i,k}$ as in the case 1). ■

Lemma 4.6. *For any integers $i > 0$, $j \ge 0$, the number K defined in* (4.31) *is positive.*

Proof: Choose n real numbers $\varrho_1 > \varrho_2 > \cdots > \varrho_{i+j}$ and denote by $\Phi_\varrho : \mathbb{R} \to \mathbb{R}^{i+j}$ an $\mathcal{E}(\varrho)$-function satisfying $\Phi_\varrho{}'(x) = ([\varrho_1]f_x, \ldots, [\varrho_1, \ldots, \varrho_{i+j}]f_x)^T$, $x \in \mathbb{R}$. Since the space $D\mathcal{E}(\varrho)$ is an $(i+j)$-dimensional extended Chebyshev space on $\mathbb{R}$, we know that $\mathrm{Osc}_j\, \Phi_\varrho(x) \cap \mathrm{Osc}_i\, \Phi_\varrho(0)$ consists of a single point for any $x < 0$. Equivalently, the function $\det\big(\Phi_\varrho{}^\sharp(x), \ldots, \Phi_\varrho{}^{\sharp\,(i-1)}(x), \Phi_\varrho{}^\sharp(0), \ldots, \Phi_\varrho{}^{\sharp\,(j-1)}(0)\big)$ keeps a strict sign on $]-\infty, 0[$. Replacing n by $i + j$ and k by 0 in the proof of Proposition 4.5, comparison of the corresponding relations (4.28) and (4.34) proves that $(-1)^{i(i-1)/2}$ and $(-1)^{i(i-1)/2} K$ have the same sign. ■

Proposition 4.7. *Suppose again that the intersection* (4.11) *consists of a single point for any $x < 0$ and $y > 0$, under the assumptions $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, $\mu_1 < \mu_2 < \cdots < \mu_n$, but now without the assumption $\{\lambda_1, \ldots, \lambda_n\} \cap \{\mu_1, \ldots, \mu_n\} = \emptyset$. Then, if $q_1(k) < \ldots < q_i(k)$ denote the i smallest integers p such that $\lambda_p \notin \{\mu_1, \ldots, \mu_k\}$, we have*

$$(-1)^{(q_1(k)-1)+\cdots+(q_i(k)-i)} \prod_{r=1}^{k}\prod_{s=1}^{i}(\lambda_{q_s(k)} - \mu_r) > 0\ . \tag{4.47}$$

Proof: The arguments are the same as in the proof of Proposition 4.5, the only change concerning the behaviour of $\delta(x)$ when $x \to -\infty$. We now have to consider

$$A := \lim_{x\to-\infty} e^{(\lambda_{q_1(k)}+\cdots+\lambda_{q_i(k)})x}\ \delta(x)\ .$$

This yields

$$A = \sum_{\{\ell_1,\ldots,\ell_i\}=\{q_1(k),\ldots,q_i(k)\}} \lambda_{\ell_2}\lambda_{\ell_3}{}^2\ldots\lambda_{\ell_i}^{i-1}\ \det\big(X_{\ell_1}, \ldots, X_{\ell_i}, \Theta(\mu_1), \ldots, \Theta(\mu_k)\big)\ ,$$

where the quantities X_ℓ are defined in (4.22). On account of (4.27), we eventually obtain

$$\mathrm{sign}(A) = (-1)^{\sum_{s=1}^{i}(q_s(k)-1)}\ \mathrm{sign}\Big(\prod_{r=1}^{k}\prod_{s=1}^{i}(\lambda_{q_s(k)} - \mu_r)\Big)\ ,$$

while (4.34) is still valid. This gives the announced result. ■

4.3. Proof of Theorem 4.1

Suppose that $\lambda_1 \geq \cdots \geq \lambda_n$, $\mu_1 \leq \cdots \leq \mu_n$. Let us first consider the case where $\{\lambda_1, \ldots, \lambda_n\} \cap \{\mu_1, \ldots, \mu_n\} = \emptyset$. Existence of blossoms in the space $\mathcal{S}(\lambda, \mu)$ means that $\mathcal{D}(\varphi)$ contains the set $\mathcal{A}$ of all n-tuples which are admissible with respect to the knot vector reduced to $(0^{[m]})$. Now, for any $x < 0$ and any $y > 0$, all n-tuples $(x^{[i]}, 0^{[j]}, y^{[k]})$ belong to $\mathcal{A}$ provided that the integer j is greater than or equal to the multiplicity m. It follows that, for any integers i, k such that $i + k \leq n - m$, the number $K_{i,k}$ defined in (4.12) is positive. Taking successively $i = 1$ and $k = 1, \ldots, n - m - 1$, then $i = 2$ and $k = 1, \ldots, n - m - 2$, and so forth up to $i = n - m - 1$ and $k = 1$, we obtain that

$$\lambda_j - \mu_i > 0 \quad \text{for all } i + j < n - m.$$

On account of the order of the λ_j's and the μ_i's, these conditions reduce to

$$\lambda_{n-m-i} - \mu_i > 0 \;, \quad 1 \leq i \leq n - m - 1 \;,$$

which is nothing but condition (3.22) of Proposition 3.14.

Let us now assume that $\{\lambda_1, \ldots, \lambda_n\} \cap \{\mu_1, \ldots, \mu_n\} \neq \emptyset$. Then, we obtain the positivity of all quantities (4.47), with $i + k \leq n - m$. We let the reader get conviced that, due to the order of the λ_s's and the μ_r's, once again this amounts to condition (3.22) of Proposition 3.14.

Finally, note that the particular case $m = 0$ gives necessary conditions for existence of blossoms in the space $\mathcal{E}(\lambda, \mu)$.

§5. Conclusions

Once and for all we fix a sequence $(\lambda^\ell)_{\ell \in \mathbb{Z}}$ of n-tuples $\lambda^\ell := (\lambda^\ell_1, \ldots, \lambda^\ell_n) \in \mathbb{R}^n$, and a sequence $(m_\ell)_{\ell \in \mathbb{Z}}$ of multiplicities. With each subdivision $\mathcal{T} := (t_\ell)_{\ell \in \mathbb{Z}}$ of the real line, let us associate the corresponding space $\mathcal{E}(\mathcal{T})$ of C^n functions on $\mathbb{R}$ with sections in the restrictions of $\mathcal{E}(\lambda^\ell)$ to $[t_\ell, t_{\ell+1}]$. Denote by $\mathcal{K}(\mathcal{T})$ the knot vector $\mathcal{K}(\mathcal{T}) := \left({t_\ell}^{[m_\ell]}\right)_{\ell \in \mathbb{Z}}$, by $\mathcal{A}(\mathcal{T})$ the corresponding set of admissible n-tuples, and by $\mathcal{S}(\mathcal{T})$ the corresponding spline space based on $\mathcal{E}(\mathcal{T})$. We can then state the following two results.

Proposition 5.1. *For each $\ell \in \mathbb{Z}$, let σ_ℓ and σ'_ℓ be two permutations of $\{1, \ldots, n\}$ such that $\lambda^{\ell-1}_{\sigma_\ell(i)} = \mu^\ell_{\sigma'_\ell(i)}$ for $i = 1, \ldots, r_\ell$, $\{\lambda^{\ell-1}_{\sigma_\ell(r_\ell+1)}, \ldots, \lambda^{\ell-1}_{\sigma_\ell(n)}\} \cap \{\lambda^\ell_{\sigma'_\ell(r_\ell+1)} \ldots, \lambda^\ell_{\sigma'_\ell(n)}\} = \emptyset$, $\lambda^{\ell-1}_{\sigma_\ell(r_\ell+1)} \geq \cdots \geq \lambda^{\ell-1}_{\sigma_\ell(n)}$ and $\lambda^\ell_{\sigma'_\ell(r_\ell+1)} \leq \cdots \leq \lambda^\ell_{\sigma'_\ell(n)}$. Suppose that, for any subdivision $\mathcal{T}$, there exists a B-spline basis in the exponential spline space $\mathcal{S}(\mathcal{T})$ and in any spline space derived from $\mathcal{S}(\mathcal{T})$ by insertion of knots. Then, we have*

$$\lambda_{\sigma_\ell(n-m_\ell-i)} - \mu_{\sigma'_\ell(r_\ell+i)} > 0, \quad 1 \leq i \leq n - m_\ell - r_\ell - 1 \;, \quad \ell \in \mathbb{Z} \;. \tag{4.48}$$

Proof: Assume the existence of B-spline bases, that is, according to Theorem 2.2, existence of blossoms in the space $\mathcal{S}(\mathcal{T})$ for any subdivision $\mathcal{T}$. For a given subdivision $\mathcal{T}$, choose a nondegenerate $\mathcal{E}(\mathcal{T})$-function $\Phi_\mathcal{T}$. Then, $\mathcal{D}(\varphi_\mathcal{T})$ contains $\mathcal{A}(\mathcal{T})$. Select an integer integer $\ell \in \mathbb{Z}$. The set $\mathcal{A}(\mathcal{T})$ contains in particular all n-tuples $(x^{[i]}, {t_\ell}^{[j]}, y^{[k]})$, $i, k \geq 1$, $j \geq m_\ell$, for any $x \in [t_{\ell-1}, t_\ell[$, and $y \in]t_\ell, t_{\ell+1}]$, and any $\ell \in \mathbb{Z}$. On account of the translation invariance of the exponential spaces, we may as well assume that $t_\ell = 0$. Denote by Φ the $\mathcal{E}(\lambda^{\ell-1}, \lambda^\ell)$-function which coincides with $\Phi_\mathcal{T}$ on $[t_{\ell-1}, t_{\ell+1}]$. The intersection

$$\mathrm{Osc}_{n-i}\, \Phi(x) \cap \mathrm{Osc}_{n-j}\, \Phi(0) \cap \mathrm{Osc}_{n-k}\, \Phi(y) = \mathrm{Osc}_{n-i}\, \Phi_\mathcal{T}(x) \cap \mathrm{Osc}_{n-j}\, \Phi_\mathcal{T}(0) \cap \mathrm{Osc}_{n-k}\, \Phi_\mathcal{T}(y) \tag{4.49}$$

therefore consists of a single point for any $(x, y) \in [t_{\ell-1}, t_\ell[\times]t_\ell, t_{\ell+1}]$ and any $i, k \geq 1$, $j \geq m_\ell$. This being valid whatever the subdivision $\mathcal{T}$, the left-hand side of (4.49) consists of a single point for any $(x, y) \in]-\infty, 0[\times]0, +\infty[$. This means existence of blossoms in the space $\mathcal{S}(\lambda^{\ell-1}, \lambda^\ell)$. Hence (4.48) results from Theorem 4.1. ■

Again, observe that the particular case of all mutiplicities equal to 0 gives a necessary condition for existence of B-spline bases in any spline space based on any space $\mathcal{E}(\mathcal{T})$, or equivalently for existence of blossoms in any space $\mathcal{E}(\mathcal{T})$. Let us consider a particular case where all knots are simple.

Proposition 5.2. *Suppose that $n = 3$ and that $m_\ell = 1$ for all $\ell \in \mathbb{Z}$. Then, the following two properties are equivalent:*

(i) *for any subdivision $\mathcal{T}$, there exists a B-spline basis in the exponential spline space $\mathcal{S}(\mathcal{T})$ and in any spline space derived from $\mathcal{S}(\mathcal{T})$ by insertion of knots,*

(ii) $\mathrm{Min}(\lambda_1^\ell, \lambda_2^\ell, \lambda_3^\ell) < \mathrm{Max}(\lambda_1^{\ell-1}, \lambda_2^{\ell-1}, \lambda_3^{\ell-1})$ *for any integer ℓ such that* $\{\lambda_1^{\ell-1}, \lambda_2^{\ell-1}, \lambda_3^{\ell-1}\} \cap \{\lambda_1^\ell, \lambda_2^\ell, \lambda_3^\ell\} = \emptyset$.

Proof: Given a subdivision $\mathcal{T}$, a triplet (x, y, z), with $x \leq y \leq z$, belongs to $\mathcal{A}(\mathcal{T})$ if and only if there exists an integer $\ell \in \mathbb{Z}$ such that either $x, y, z \in [t_\ell, t_{\ell+1}]$ or $x \in [t_{\ell-1}, t_\ell[$, $y = t_\ell$, and $z \in]t_\ell, t_{\ell+1}]$. Therefore, requiring existence of blossoms in the spline space $\mathcal{S}(\mathcal{T})$ for any subdivision $\mathcal{T}$ is now equivalent to requiring existence of blossoms in each space $\mathcal{S}(\lambda^{\ell-1}, \lambda^\ell)$ with multiplicity one. Therefore (ii) follows from Theorem 4.1. ■

Conjecture. The necessary condition stated in the Proposition 5.1 becomes sufficient too in the particular case considered in Proposition 5.2. We conjecture that it is always necessary and sufficient. However proving the sufficiency (either by considering intersections of osculating flats, or by limiting the number of zeros in the space $\mathcal{E}(\mathcal{T})^\sharp$ by modifying P.J. Barry's results) is certainly not easy in the general case.

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