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Preconditioned Iterative Methods for Scattered Data Interpolation

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The scattered data interpolation problem in two space dimensions is formulated as a partial differential equation with interpolating side conditions. The system is discretized by the Morley finite element space. The focus of this paper is to study preconditioned iterative methods for the corresponding discrete systems. We introduce block diagonal preconditioners, where a multigrid operator is used for the differential equation part of the system, while we propose an operator constructed from thin plate radial basis functions for the equations corresponding to the interpolation conditions. The effect of the preconditioners are documented by numerical experiments.

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1. Introduction

We consider a scattered data interpolation problem where $\{x_i\}_{i=1}^m$ is a finite number of interpolation points in an open subset Ω of $\mathbb{R}^2$, and $\{g_i\}_{i=1}^m$ is the set of corresponding real interpolation values. For a general introduction to such problems see [21].

Throughout this paper $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain and $\partial\Omega$ is the boundary. Furthermore,, define an "energy functional", $E : H_0^2 \mapsto \mathbb{R}$, by

$$E(v) = \sum_{|\alpha|=2} \int_\Omega |D^\alpha v|^2 \, dx,$$

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where $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}$ denotes a partial derivative of total order $|\alpha| = \alpha_1 + \alpha_2$. The linear space $H_0^2 = H_0^2(\Omega)$ is the Sobolev space

$$H_0^2 = \{v \in L^2 \, : \, D^\alpha v \in L^2, \; |\alpha| \le 2, \; v = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega\}.$$

Here n is the outward normal vector on $\partial\Omega$.

The purpose of this paper is to study the following constrained minimization problem:

$$\min_{v \in H_0^2} E(v) \quad \text{subject to } v(x_i) = g_i, \quad i = 1, 2, \ldots, m. \tag{1.1}$$

This problem has a unique solution $u \in H_0^2$. If the interpolation points, $\{x_i\}$, are allowed to be more or less arbitrary located in Ω, with no obvious structure, then the problem (1.1) is usually referred to as a scattered data interpolation problem. We have formulated homogeneous boundary conditions. Extension to nonhomogeneous boundary conditions is straightforward.

We observe that the corresponding problem in one dimension is simply the wellknown classical cubic spline interpolation problem. This one dimensional interpolation problem is closely tied to the corresponding biharmonic operator, i.e. the fourth derivative operator with Dirichlet boundary conditions. In the analysis of the classical cubic spline interpolation problem the fact that the Green's function of the biharmonic operator is a simple piecewise cubic C^2–function is heavily utilized. However, in higher dimensions the corresponding Green's function, with respect to the domain Ω, will, in general, not have such a simple structure. In this case it is therefore not possible to reduce the problem (1.1) into a simpler interpolation problem, where the unknown function u is a linear combination of m a priori prescribed basis functions.

Of course, there are some exceptions. For example, if Ω is a circular disc then the solution of the Dirichlet problem for the biharmonic equation, with a point source at the center, is a rotation invariant function which can be explicitly given. Linear combinations of such *radial basis functions* are frequently referred to as *thin plate splines*, due to the fact that the biharmonic equation models the deformation of infinitely thin plates under given loads ([9]). These functions have frequently been used in practical scattered data interpolation problems. vThe main idea is to construct an interpolating function which is a linear combination of the radial basis functions. However, in contrast to the one dimensional case, the obtained solution will not correspond to an optimal solution of (1.1). In particular, it is usually not possible to have the thin plate splines satisfy the prescribed boundary conditions.

Let Δ denote the Laplace operator. The solution u of the constrained minimization problem (1.1) can be alternatively characterized as the solution of the following differential–interpolation system of saddle point structure:

$$\begin{pmatrix} \Delta^2 & \pi^* \\ \pi & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{g} \end{pmatrix}. \tag{1.2}$$

Here the extra unknown vector, $\lambda \in \mathbb{R}^m$, corresponds to Langrange multipliers associated with each constraint. These multipliers correspond physically to the forces required to keep "the plate" in the given position. The operator $\Delta^2 : H_0^2 \mapsto H^{-2} = (H_0^2)^*$ is the biharmonic operator. More precisely, H^{-2} is the space of bounded linear functionals on H_0^2, and

$$\langle \Delta^2 u, v\rangle = \int_\Omega (\Delta u)\,(\Delta v)\,dx,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between H^{-2} and H_0^2. In other words, for any linear functional $f \in (H_0^2)^* = H^{-2}$, and any $v \in H_0^2$, $\langle f, v\rangle \in \mathbb{R}$ is simply the result of f applied to v. The operator $\pi : H_0^2 \mapsto \mathbb{R}^m$ is given by

$$(\pi v)_i = v(x_i),$$

while the adjoint operator $\pi^* : \mathbb{R}^m \mapsto H^{-2}$ is given by

$$\langle \pi^* \lambda, v\rangle = \sum_{i=1}^m \lambda_i v(x_i), \quad \forall v \in H_0^2.$$

In order to obtain approximate solutions of (1.1), or equivalently (1.2), in higher dimensions one is forced to perform a suitable numerical discretization of the problem. In the present paper we shall study a finite element discretization of the problem, where the space H_0^2 is replaced by a finite element space. An obvious drawback with such an approach is that we are lead to large, ill–conditioned linear systems. In fact, a corresponding discretization of the biharmonic operator is just a block of the complete coefficient matrix of the discrete system, and such operators are well known to be very ill–conditioned when the mesh is fine. In addition, the conditioning of the coefficient matrix can be strongly affected by the location of the interpolation points, and the coefficient matrix is in general symmetric, but indefinite. The main purpose of the present paper is to study preconditioned iterative methods for the systems obtained from a finite element discretization of (1.2). For a discussion of iterative methods and preconditioning for radial basis function approaches to scattered data interpolation see [3,4,11, 19,20].

The discretization of the system (1.2) is based on the nonconforming Morley finite element space and is presented in §2, while we give a brief review of preconditioned iterative methods for linear systems of saddle point structure in §3. Based on this discussion we motivate our choice of preconditioners for the discrete versions of the system (1.2) in §4. Finally, we document the effect of the preconditioners in §5, where a collection of numerical experiments is presented.

2. Preliminaries

For $k \geq 0$ we will use $H^k = H^k(\Omega)$ to denote the Sobolev space of functions mapping Ω to $\mathbb{R}$ with k derivatives in $L^2 = L^2(\Omega)$, with corresponding norm given by

$$\|u\|_k^2 = \sum_{|\alpha|\leq k} \|D^\alpha u\|_0^2 = \sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^2 \, dx.$$

The space $H_0^k = H_0^k(\Omega)$ will denote the closure in H^k of $C_0^\infty(\Omega)$, and the dual space of H_0^k with respect to the L^2 inner product will be denoted by H^{-k}. Hence, $L^2 \subset H^{-k}$ and for any $f \in L^2$ and $v \in H_0^k$ the duality pairing $\langle f, v\rangle$ is given by the L^2 inner product, i.e.

$$\langle f, v\rangle = \int_\Omega f v \, dx. \tag{2.1}$$

In fact, we will also use $\langle \cdot, \cdot \rangle$ to denote the inner product in L^2, i.e. $\langle f, v\rangle$ is given by (2.1) for any $f, v \in L^2$. For a general review of Sobolev spaces we referer to [1].

2.1. The Biharmonic Equation

Consider the biharmonic equation with homogeneous Dirichlet boundary conditions given by:

$$\begin{aligned} \Delta^2 u = f &\qquad \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 &\qquad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

The biharmonic equation models the transverse displacement of a thin plate which is clamped at the boundary. The function f represents the transverse forces acting on the plate.

Let $a : H_0^2 \times H_0^2 \mapsto \mathbb{R}$ denote the bilinear form

$$a(u, v) = \sum_{|\alpha|=2} \int_\Omega (D^\alpha u)(D^\alpha v) \, dx. \tag{2.3}$$

This bilinear form is positive definte on H_0^2. On the space H_0^2 the form a can be equivalently written in a simplified form as

$$a(u, v) = \int_\Omega (\Delta u)(\Delta v) \, dx. \tag{2.4}$$

However, this identity is not true for all functions in H^2.

For any $f \in H^{-2}$ the weak formulation of (2.2) is given by:

Find $u \in H_0^2$ such that

$$a(u, v) = \langle f, v\rangle \qquad \text{for } v \in H_0^2. \tag{2.5}$$

With this notation the differential–interpolation system (1.2) can alternatively be written in the following symmetric weak form:

Find $(u,\lambda) \in H_0^2 \times \mathbb{R}^m$ such that

$$\begin{aligned} a(u,v) + \lambda \cdot \pi v &= 0 && \text{for } v \in H_0^2, \\ \pi u \cdot \mu &= \mu \cdot g && \text{for } \mu \in \mathbb{R}^m, \end{aligned} \tag{2.6}$$

where $\lambda \cdot \mu$ denotes the usual Eucledian inner product in $\mathbb{R}^m$. It is straightforward to verify, for example by using the saddle point theory of [7], that this system has a unique solution, and that the u–component of this solution also solves the constrained minimization problem (1.1).

Our discretization of the problem (1.1) will be based on the weak fomulation (2.6). However, first we will describe a discretization of the biharmonic problem (2.2), or more precisely (2.5). The positive definite system (2.5) can, in theory, be discretized by a standard finite element approach, cf. for example [6] or [8]. However, a technical difficulty is that for a conforming finite element method we are required to use subspaces of H_0^2. In order to construct such a piecewise polynomial space in two space dimensions we either have to use polynomials of degree of at least five, or alternatively use macro elements like the Clough–Tocher space ([6,8]). Therefore, in our experiments above we have used a slightly simpler nonconforming approximation.

Let $\mathcal{T}_h$ be a triangulation of Ω, where h is a discretization parameter indicating the size of the triangles. The Morley space, V_h, corresponding to $\mathcal{T}_h$ consists of piecewise quadratics, which are continuous at each vertex, and where the normal derivative is continuous at the midpoint of each edge. The degrees of freedom for a function $v \in V_h$ is the value of v at each vertex and the normal derivative of v at the midpoint of each edge, cf. Figure 1. Furthermore, in order to approximate the Dirichlet boundary conditions, degrees of freedom associated with boundary vertices or edges are required to be zero.

The space V_h is not a subspace of $C^1(\Omega)$, and therefore not a subspace H_0^2. As a consequence, the space will not lead to a conforming discretization of the biharmonic problem (2.5). In fact, the elements of V_h are not continuous, and therefore V_h is not even a subspace of H^1. However, the functions in V_h are sufficiently weakly continuous such that the space can be used to define a nonconforming discretization of (2.5).

Define a bilinear form $a_h : V_h \times V_h \mapsto \mathbb{R}$ by

$$a_h(u,v) = \sum_{T \in \mathcal{T}_h} \sum_{|\alpha|=2} \int_T (D^\alpha u)(D^\alpha v)\, dx.$$

This bilinear form is a discretization of the form a introduced above. However, since $V_h \not\subseteq H_0^2$ the form a_h is defined piecewise on each triangle. It can easily be verified that the bilinear form a_h is positive definite on V_h.

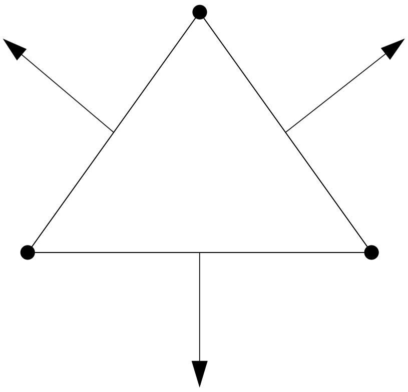

Figure 1. The six degrees of freedom of the Morley element.

Assume that $f \in L^2$. The discrete approximation of the biharmonic problem (2.2) is given by:

Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \langle f, v \rangle \qquad \text{for } v \in V_h. \tag{2.7}$$

The properties of the Morley method (2.7), as an approximation of the biharmonic problem (2.5), is well documented in the literature. For example, under proper conditions we obtain linear convergence with respect to h in the energy norm derived from the form a_h. For discussions on the Morley method applied to the biharmonic equation, and similar fourth order equations, we refer to [2,10,14,22].

2.2. The Discrete Interpolation Problem

When the continuous space H_0^2 is replaced by the finite element space V_h the corresponding approximation of the interpolation problem (1.1) is given by:

$$\min_{v \in V_h} E_h(v) \quad \text{subject to } v(x_i) = g_i, \quad i = 1, 2, \ldots, m, \tag{2.8}$$

where $E_h(v) = a_h(v, v)$.

In order to ensure that this discrete interpolation problem has a unique solution for arbitrary data g we must require that the finite element space V_h and the interpolation points $\{x_i\}$ satisfies the following rank condition:

$$\begin{cases} \text{For each } g \in \mathbb{R}^m \text{ there exists at least one } v \in V_h \\ \text{such that } v(x_i) = g_i, \qquad i = 1, 2, \ldots, m. \end{cases} \tag{2.9}$$

If this condition is satisfied then obviously no triangle of $\mathcal{T}_h$ can contain more than six interpolation points. On the other hand, if the interpolation points are fixed, then we can always make the triangulation $\mathcal{T}_h$ sufficiently fine such that (2.9) holds. In our discussion below we will always assume that this is the case.

If condition (2.9) holds then there is a unique solution $u_h \in V_h$ of the discrete interpolation problem (2.8). Furthermore, this solution can be obtained from a discrete analog of (2.6) given by:

Find $(u_h, \lambda_h) \in V_h \times \mathbb{R}^m$ such that

$$\begin{aligned} a_h(u_h, v) + \lambda_h \cdot \pi_h v &= 0 \qquad && \text{for } v \in V_h, \\ \pi_h u_h \cdot \mu &= \mu \cdot g \qquad && \text{for } \mu \in \mathbb{R}^m. \end{aligned} \tag{2.10}$$

Here, $\pi_h : V_h \mapsto \mathbb{R}^m$ is the discrete analog of π, i.e.

$$(\pi_h v)_i = v(x_i).$$

This system is a symmetric, but indefinite, linear system. Furthermore, condition (2.9) implies that the system is nonsingular. Of course, if the triangulation $\mathcal{T}_h$ is sufficiently fine, then the system (2.10) is large and ill–conditioned. Therefore, the main purpose of this paper is to design effective preconditioners for this system.

Define a discrete biharmonic operator $\Delta_h^2 : V_h \mapsto V_h$ by

$$\langle \Delta_h^2 v, w \rangle = a_h(v, w) \qquad \text{for all } v, w \in V_h,$$

where we recall that $\langle \cdot, \cdot \rangle$ is the L^2 inner product. This is in fact a slight abuse of notation since Δ_h^2 is not a square of a locally defined operator Δ_h. However, the operator Δ_h^2 is easily seen to be an L^2 symmetric and positive definite operator on V_h. Define also $\pi_h^* : \mathbb{R}^m \mapsto V_h$ as the adjoint of π_h, i.e.

$$\langle \pi_h^* \mu, v \rangle = \mu \cdot \pi_h v \qquad \text{for all } \mu \in \mathbb{R}^m, v \in V_h.$$

By using these operators the discrete system (2.10) can be equivalently written in an operator form, similar to (1.2) as

$$\begin{pmatrix} \Delta_h^2 & \pi_h^* \\ \pi_h & 0 \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{g} \end{pmatrix}. \tag{2.11}$$

When the mesh parameter h tends to zero the condition number of the discrete biharmonic operator, Δ_h^2, blows up like h^{-4}, causing the system (2.11) to be ill–conditioned. However, in practical applications the number of interpolation points m will also be large. This will cause additional ill–conditioning effects, in particular if the interpolation points are clustered or if points are located close to the boundary.

3. Iterative Methods and Preconditioning

If the triangulation $\mathcal{T}_h$ is sufficiently fine and the number of interpolation points is large the system (2.11) can, in practise, not be solved by a direct method. Therefore, we are forced to consider iterative methods. The symmetric saddle point system (2.11) can in theory be solved by a Krylov space method, like the conjugate residual method or the minimimum residual method, which are methods designed for symmetric, but not necessary positive definite, systems. Alternatively, the standard conjugate gradient method can be applied to the normal equations. For a general review of iterative methods, in particular Krylov space methods, we refer for example to [12,18]. A more thorough discussion of the use of iterative methods for symmetric systems, with a saddle point structure, can be found in [13,16,17,23].

Let $\mathcal{A}_h : V_h \times \mathbb{R}^m \mapsto V_h \times \mathbb{R}^m$ denote the coefficient operator of the system (2.11), i.e.

$$\mathcal{A}_h = \begin{pmatrix} \Delta_h^2 & \pi_h^* \\ \pi_h & 0 \end{pmatrix}.$$

Since the condition number of this operator gets extremely large as the mesh parameter h gets small, we can not apply an iterative method directly to the system (2.11). The introduction of a suitable preconditioner is necessary in order to obtain a system with a moderate condition number. This preconditioner has to address both the fine mesh effects and the bad conditioning created by the interpolation points. In this paper we shall only consider block diagonal, symmetric and positive definite preconditioners $\mathcal{B}_h : V_h \times \mathbb{R}^m \mapsto V_h \times \mathbb{R}^m$ of the form

$$\mathcal{B}_h = \begin{pmatrix} M_h & 0 \\ 0 & N_h \end{pmatrix}.$$

Hence, the preconditioned system takes the form

$$\mathcal{B}_h \mathcal{A}_h \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \mathcal{B}_h \begin{pmatrix} 0 \\ \mathbf{g} \end{pmatrix}. \tag{3.1}$$

Our discussion below follows a common approach to preconditioned iterative methods for saddle point problems as outlined in [13,16,17,23].

We note that the coefficient operator, $\mathcal{B}_h\mathcal{A}_h$, of the preconditioned system is symmetric with respect to inner product generated by $\mathcal{B}_h^{-1}$, and the convergence rate of any Krylov iterative method can be bounded with respect to the condition number of this operator. Furthermore, it is well known, cf. for example [16], that the condition number of the operator $\mathcal{B}_h\mathcal{A}_h$ can be controlled by the condition number of the operators

$$M_h \Delta_h^2 \qquad \text{and} \qquad N_h(\pi_h \Delta_h^{-2} \pi_h^*).$$

Here Δ_h^{-2} denotes the inverse of Δ_h^2. Our preconditioning strategy is therefore to choose $M_h : V_h \mapsto V_h$ and $N_h : \mathbb{R}^m \mapsto \mathbb{R}^m$ as symmetric positive definite operators such that

$$M_h \approx \Delta_h^{-2} \qquad \text{and} \qquad N_h \approx (\pi_h \Delta_h^{-2} \pi_h^*)^{-1}. \tag{3.2}$$

We observe, in particular, that if $M_h = \Delta_h^{-2}$ and $N_h = (\pi_h \Delta_h^{-2} \pi_h^*)^{-1}$ then

$$\mathcal{B}_h \mathcal{A}_h = \begin{pmatrix} I & \Delta_h^{-2} \pi_h^* \\ (\pi_h \Delta_h^{-2} \pi_h^*)^{-1} \pi_h & 0 \end{pmatrix} \equiv \begin{pmatrix} I & Q_h^* \\ Q_h & 0 \end{pmatrix}. \tag{3.3}$$

Here, $Q_h : V_h \mapsto \mathbb{R}^m$ and $Q_h^* : \mathbb{R}^m \mapsto V_h$ are adjoint operators with respect to the inner product induced by $\mathcal{B}_h^{-1}$, i.e.

$$a_h(u,v) \quad \text{on } V_h \quad \text{and} \quad (\pi_h \Delta_h^{-2} \pi_h^*)\lambda \cdot \mu \quad \text{on } \mathbb{R}^m.$$

Furthermore, the eigenvalues of the operator $\mathcal{B}_h \mathcal{A}_h$ are contained in the set of three points given by $\{1, (1 \pm \sqrt{5})/2\}$. Hence, our goal is to design efficient preconditioners which nearly reach this behavior.

4. The Construction of Preconditioners

Following (3.2) the preconditioner M_h, defined on V_h, should be a good preconditioner for the discrete biharmonic operator Δ_h^2. The construction of such multigrid preconditioners, M_h, has been discussed by several authors, cf. for example [5] and references given there. It is established, that under proper conditions these preconditioners lead to condition numbers for the operator $M_h \Delta_h^2$ which are bounded uniformly in h. Hence, we will not consider this issue any further.

The main contribution here is therefore to propose suitable preconditioners $N_h : \mathbb{R}^m \mapsto \mathbb{R}^m$ such that $N_h \approx (\pi_h \Delta_h^{-2} \pi_h^*)^{-1}$. Since the preconditioner N_h is a discrete operator, mapping discrete functions defined at the interpolation points into itself, we will refer to N_h as a *point preconditioner*.

Ideally, we would like to construct N_h such that the condition number of $N_h(\pi_h \Delta_h^{-2} \pi_h^*)$ is independent of the mesh parameter h, independent of the number of interpolation points m, and independent of the location of the interpolation points. In addition, an evaluation of the operator N_h must be cheap. As will appear clear from the discussion below, at the moment we are far from being able to show theoretically that the preconditioners we propose reach this ideal property. On the other hand, the experiments given in the next section will clearly indicate that the preconditioners we propose drastically reduce the conditon number of the coefficient operator and speed up the convergence when Krylov space methods are applied to the discrete system (2.11).

The discussion on the construction of the operator N_h given below is unaffected by the finite element discretization. In fact, we will always choose N_h to be independent of h, i.e. $N_h = N$. Therefore, for the moment, we forget the space discretization and return to the continuous system (1.2). Hence, we would like to construct a symmetric and positive definite operator $N : \mathbb{R}^m \mapsto \mathbb{R}^m$ such that $N \approx (\pi \Delta^{-2} \pi^*)^{-1}$, where Δ^{-2} denotes the solution operator of the biharmonic problem (2.2).

Consider the system (1.2), or equivalently (2.6). It is straightforward to check that the mapping $g \mapsto \lambda$ is given by

$$\lambda = -(\pi \Delta^{-2} \pi^*)^{-1} g.$$

In other words, the desired point preconditioner N should approximate the solution map $g \mapsto -\lambda$ of the system (1.2). Furthermore, note that

$$(\pi \Delta^{-2} \pi^*)^{-1} g \cdot g = -\lambda \cdot g = -\lambda \cdot \pi u = a(u,u) = \min_{\substack{v \in H_0^2 \\ v(x_i) = g_i}} E(v). \tag{4.1}$$

Hence, for each $g \in \mathbb{R}^m$ the bilinear form $(\pi \Delta^{-2} \pi^*)^{-1} g \cdot g$ is equal to the corresponding minimum value of $E(u)$ obtained from (1.1).

4.1. The Explicit Point Preconditioner

We first construct a simple explicit point preconditioner, i.e. we shall propose an operator $N : \mathbb{R}^m \mapsto \mathbb{R}^m$ with an explicitly given matrix representation. Consider the problem (1.1) with two closely located interpolation points x_1 and x_2 far from the boundary $\partial\Omega$. Then the mapping $g \mapsto \lambda = (\pi \Delta^{-2} \pi^*)^{-1} g$ an be roughly estimated as

$$\lambda_1 = c_1 d^{-2}(g_1 - g_2), \quad \lambda_2 = c_1 d^{-2}(g_2 - g_1),$$

where d is the distance between the points and $c_1 > 0$ is a constant. This formula can either be found from numerical experiments or it can be motivated from the fundamental solutions presented in the next section.

Alternatively, if we have a single interpolation point close to the boundary then operator $g \mapsto \lambda = (\pi \Delta^{-2} \pi^*)^{-1} g$ can be estimated as

$$\lambda_1 = c_2 d^{-2} g_1$$

where d is the distance to the boundary and $c_2 > 0$ is a constant.

If we assume that, in a distribution with many points, we can add the forces into the effects of all pairs of points and the boundary contributions, we obtain the formula:

$$\lambda_i = \sum_{j=1}^{m} c_1 d_{ij}^{-2}(g_i - g_j) + c_2 d_i^{-2} g_i, \qquad i = 1, 2, \ldots, m. \tag{4.2}$$

Here d_{ij} is the distance between the two interpolation points x_i and x_j, d_i is the distance from the point x_i to the boundary, and c_1, c_2 are positive proportionality constants. Our explicit point preconditioner is the map $g \mapsto \lambda = Ng$, where λ is given by (4.2). It is straightforward to check that this operator is symmetric and positive definite.

4.2. The Implicit Point Preconditioner

In contrast to the explicit preconditioner presented above the implicit point preconditioner, which we now shall describe, will require the inversion or, more precisely, an LU decomposition of an $m \times m$ matrix. However, we are only required to do this computation once for each configuration of interpolation points. Hence, this LU decomposition can be considered as "overhead." Also observe that when the triangulation $\mathcal{T}_h$ is fine, the dimension of the discrete system (2.11) is much larger than m.

In order to construct the implicit preconditioner we shall use the fundamental solution of the biharmonic operator Δ^2 with respect to circular discs. Let $d > 0$ and consider the problem (1.1) with $\Omega = B_d = \{x : |x| \le d\}$, with a single interpolation point at the origin, and with required interpolation value one. The solution $\phi = \phi(\cdot, d)$ can be characterized by the following boundary value problem, cf (1.2),

$$\begin{aligned} \Delta^2\phi &= 0 \qquad \text{for } x \in B_d \setminus \{0\}, \\ \phi = \frac{\partial \phi}{\partial n} &= 0 \qquad \text{for } x \in \partial B_d, \\ \phi(0) &= 1. \end{aligned}$$

The solution of this problem is rotation invariant, and given by

$$\phi(r) = \phi(r, d) = \left(\frac{r}{d}\right)^2 (2 \log \frac{r}{d} - 1) + 1, \tag{4.3}$$

where r denotes the distance to the origin.

For each interpolation point x_i define $\phi_i \in H_0^2(\Omega)$ by

$$\phi_i(x) = \begin{cases} \phi(x - x_i, d_i) & \text{for } |x - x_i| \le d_i, \\ 0 & \text{for } |x - x_i| > d_i, \end{cases}$$

where, as above, d_i denote the distance from x_i to $\partial\Omega$. Hence, the function ϕ_i is identically zero on the complement of the ball Ω_i, where $\Omega_i \subset \Omega$ is given by

$$\Omega_i = \{x : |x - x_i| < d_i\}.$$

Define $W = W_X \subset H_0^2(\Omega)$ as the span of $\phi_1, \phi_2, \ldots, \phi_m$, where the subscript X indicates the dependence on the set of interpolation points. Hence, W_X is an m dimensional subspace of $H_0^2(\Omega)$.

The implicit preconditioner will be constructed by approximating the original problem (1.1) by a Galerkin approximation, where the function class is reduced from all of $H_0^2(\Omega)$ to W. In order to do this we have to assume that the space W satisfies a rank condition analog to (2.9). More precisely, we assume that

$$\begin{cases} \text{for each } g \in \mathbb{R}^m \text{ there exists a } w \in W \\ \text{such that } w(x_i) = g_i, \qquad i = 1, 2, \ldots, m. \end{cases} \tag{4.4}$$

We note that since dim $W = m$ this is equivalent to assume that the $m \times m$ matrix $\phi_i(x_j)$ is nonsingular, or that the map $\pi_X : \mathbb{R}^m \mapsto \mathbb{R}^m$ is invertible. Here $\pi_X = \pi|_W$. We shall not discuss the condition (4.4) any further here. In our numerical experiments we have just assumed that this condition holds, and this has not caused any computational problems as long as the mesh is sufficiently fine.. However, some theoretical discussions on this issue can be found in [15].

Consider now the constrained minimization problem:

$$\min_{w \in W} E(w) \quad \text{subject to } w(x_i) = g_i, \quad i = 1, 2, \dots, m. \tag{4.5}$$

Exactly as before we can argue that the solution $u_X \in W$ of this problem is characterized by a saddle point system. In weak form this system can be written:

Find $(u_X, \lambda_X) \in W \times \mathbb{R}^m$ such that

$$\begin{aligned} a(u_X, w) + \lambda_X \cdot \pi_X w &= 0 \qquad &&\text{for } w \in W, \\ \pi_X u_X \cdot \mu \qquad\qquad &= \mu \cdot g &&\text{for } \mu \in \mathbb{R}^m. \end{aligned} \tag{4.6}$$

The implicit point preconditioner $N : \mathbb{R}^m \mapsto \mathbb{R}^m$ is defined as the map $g \mapsto -\lambda_X$, or in operator form, $N = (\pi_X A_X^{-1} \pi_X^*)^{-1}$. Here $\pi_X^* : \mathbb{R}^m \mapsto W$ is the dual operator of π_X, while $A_X : W \mapsto W$ is the corresponding descrete biharmonic operator, i.e. A_X has a matrix representation $a(\phi_i, \phi_j)$ with respect to the basis $\{\phi_i\}$ of W.

Note that, since π_X and A_X both are invertible, the preconditioner N can alternatively be written as

$$N = \pi_X^{-*} A_X \pi_X^{-1}. \tag{4.7}$$

Hence, an evaluation of N is cheap as long as an LU decomposition of the matrix $\phi_i(x_j)$ is precomputed.

We recall from (3.2) that our goal is to construct the preconditioner N such that $N \approx (\pi \Delta^{-2} \pi^*)^{-1}$ or, more precisely, such that the condition number of $N(\pi \Delta^{-2} \pi^*)$ is small. Hence, we would like to estimate the extreme eigenvalues of this operator. In the same way as we derived (4.1) from the system (1.2), or (2.6), we obtain from the system (4.6) above that

$$Ng \cdot g = -\lambda_X \cdot g = -\lambda_X \cdot \pi_X u_X = E(u_X).$$

Furthermore, since $W \subset H_0^2$ we have $E(u) \le E(u_X)$ or

$$Ng \cdot g \ge (\pi \Delta^{-2} \pi^*)^{-1} g \cdot g \qquad \text{for } g \in \mathbb{R}^m.$$

As a consequence we obtain that the minimum eigenvalue of $N(\pi \Delta^{-2} \pi^*)$ is bounded below by one. However, it is much harder to derive a sharp upper bound for the spectrum of $N(\pi \Delta^{-2} \pi^*)$. Such a bound is usually derived from an inequality of the form

$$Ng \cdot g \le c\, (\pi \Delta^{-2} \pi^*)^{-1} g \cdot g \qquad \text{for } g \in \mathbb{R}^m,$$

where the positive constant c will be the desires upper bound. However, we will make no attempt to derive such a bound here.

Remark. In order to evaluate the preconditioner $N = \pi_X^{-*} A_X \pi_X^{-1}$ we have to compute the matrix representation of the operator A_X. Hence, we need to compute the integrals

$$a(\phi_i, \phi_j) = \int_{\Omega_i \cap \Omega_j} (\Delta\phi_i)(\Delta\phi_j)\, dx, \tag{4.8}$$

where we have used (2.4) to simplify the bilinear form. It seems hard to evaluate these integrals analytically. Therefore, we will use numerical integration. However, the function ϕ_i has a weak singularity at the point x_i. Therefore, we use Green's theorem to obtain

$$\begin{aligned} a(\phi_i, \phi_j) &= \int_{\Omega_i \cap \Omega_j} (\Delta\phi_i)(\Delta\phi_j)\, dx \\ &= \int_{\Omega_i \cap \Omega_j} (\Delta^2\phi_i)\phi_j\, dx - \int_\Gamma \frac{\partial \Delta\phi_i}{\partial n}\phi_j\, ds + \int_\Gamma \Delta\phi_i \frac{\partial\phi_j}{\partial n}\, ds. \end{aligned}$$

Here $\Gamma = \partial\Omega_i \cap \Omega_j$. Since ϕ_i is a fundamental solution, with a known singularity at x_i, the first integral can be given exatly as

$$\int_{\Omega_i \cap \Omega_j} (\Delta^2\phi_i)\phi_j\, dx = \frac{16\pi}{d_i^2}\phi_j(x_i),$$

while the line integrals are approximated by one dimensional integration procedures.

5. Numerical Experiments

As we have indicated above we will not give a full theoretical analysis of the preconditioners we have proposed above. Instead, we will present a series of numerical experiments in order to illustrate the effect of these preconditioners. The discrete system (2.11) is solved by a preconditioned iterative method, and different preconditioners will be tested. The domain Ω is kept fixed throughout, while the configuration of the interpolation points and the mesh varies. The domain Ω is taken to be the unit square. The domain is triangulated by first dividing it into squares of size $h \times h$. Then each square is divided into two triangles by the diagonal with negative slope to obtain the triangulation $\mathcal{T}_h$.

We will use a block diagonal preconditioner $\mathcal{B}_h = \text{diag}(M_h, N_h)$. The preconditioner M_h, acting on the Morley space V_h, will, in all experiments, be a multigrid preconditioner of the form described in [5]. As a point preconditioner $N_h : \mathbb{R}^m \mapsto \mathbb{R}^m$ we will either use the identity (i.e. no preconditioner), a simple diagonal scaling operator, or one of the two preconditioners described above.

For the explicit point preconditioner the proportionality constans are chosen as $c_1 = 12$ and $c_2 = 25$.

In the experiments we report the required number of iterations, and estimates for the condition number of the coefficient operator, $\mathcal{B}_h\mathcal{A}_h$, of the preconditioned system (3.1). As a stopping criteria for the iterations we require that the residual, in the norm generated by $\mathcal{B}_h^{-1}$, have been reduced by a given tolerance ϵ. More precisely, we require

$$\frac{(\mathcal{B}_h^{-1}r^n, r^n)}{(\mathcal{B}_h^{-1}r^0, r^0)} \leq \epsilon,$$

where $r^n \in V_h \times \mathbb{R}^m$ is the residual after n iterations, and $(\cdot,\cdot)$ denotes the inner product on $V_h \times \mathbb{R}^m$ induced by the L^2 inner product on V_h and the Eucledian inner product on $\mathbb{R}^m$.

The spectral condition number of the symmetric indefinte operator $\mathcal{B}_h\mathcal{A}_h$ is given by

$$\kappa(\mathcal{B}_h\mathcal{A}_h) = \frac{\sup|\sigma|}{\inf|\sigma|},$$

where the supremum and infimium is taken over the spectrum of $\mathcal{B}_h\mathcal{A}_h$. In the experiments below the condition numbers are estimated by applying the conjugate gradient method to the normal system, i.e. we solve:

$$(\mathcal{B}_h\mathcal{A}_h)^2\begin{pmatrix}u_h\\ \lambda_h\end{pmatrix} = (\mathcal{B}_h\mathcal{A}_h)\mathcal{B}_h\begin{pmatrix}0\\ \mathbf{g}\end{pmatrix} \tag{5.1}$$

and using standard techniques for estimating $\kappa((\mathcal{B}_h\mathcal{A}_h)^2) = (\kappa(\mathcal{B}_h\mathcal{A}_h))^2$. When we use the conjugate gradient method applied to the normal equation as an iterative method we will count each iteration as two iterations. This is because each iteration for the normal system require two evaluations of the operator $\mathcal{B}_h\mathcal{A}_h$, in contrast to the conjugate residual method. Hence, the iteration counts for the two methods represents comparable quantities.

In all our examples we have chosen the interpolation values g_i to alternate between 1 and -1.

Example 5.1

In this example we use uniformly distributed interpolation points, see Figure 2, and we only use the standard conjugate gradient method applied to the normal system as a solver. As an initial experiment we try to use no point preconditioner, i.e. $N_h = I$. Hence, only the biharmonic multigrid preconditioner is used. The tolerance ϵ is chosen as 10^{-5}. The results are given in Table 1, where κ is the estimated value for $\kappa(\mathcal{B}_hA_h)$. As explained above, the reported number of iterations are the double of the actual conjugate gradient iterations.

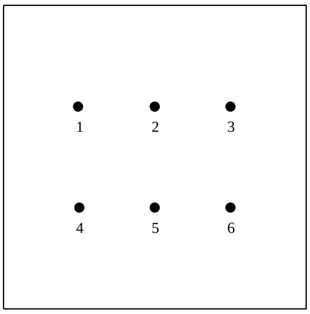

Figure 2. An example with 6=3×2 uniformly distributed interpolation points.

h \ m	1		4=2×2		9=3×3	
	κ	# it	κ	# it	κ	# it
2^{-3}	160	16	418	40	970	40
2^{-4}	223	22	642	26	6594	132
2^{-5}	279	28	1011	32	3240	172
2^{-6}	309	32	839	38	3487	192

Table 1
The condition numbers and the number of required iterations with the preconditioned conjugate gradient method for the normal system. $N_h = I$.

We observe from Table 1 that the condition numbers grow quickly with increasing numer of interpolation points. The convergence is in fact very unstable and if we use more than ten interpolation points the algorithm will seldom converge.

We also observe that the condition numbers are increasing much faster than the required number of iterations. This is an indication of a scaling problem. Therefore, we will try a simple diagonal point preconditioner given by

$$\lambda_i = \frac{16 * \pi}{d_i^2} g_i,$$

where d_i is the distance from interpolation point x_i to the boundary. This diagonal preconditioner can be motivated from the fundamental solutions presented in Section 4. The results are given in Table 2.

By comparing Table 1 and Table 2 we observe that the condition numbers have been reduced dramatically by introducing the diagonal point preconditioner, and also the iteration counts are clearly improved. However, the convergence is still unstable when the number of interpolation points increase.

We should also remark that the estimates for the condition numbers in these tests are not very reliable. The reason for this is that the tolerance ϵ is too large and that the convergence is rather unstable. But this example clearly motivates

m	1		9=3×3		25=5×5		49=7×7	
h	κ	# it	κ	# it	κ	# it	κ	# it
2^{-3}	2.3	10	10.5	24	31.2	92	64.5	182
2^{-4}	2.6	12	12.7	30	37.9	122	79.5	208
2^{-5}	2.8	14	14.3	36	44.4	106	99.5	248
2^{-6}	2.8	16	14.1	38	46.4	98	111.2	276

Table 2
The condition numbers and the required number of iterations with a diagonal point preconditioner.

that there is a need for efficient point preconditioners. These preconditioners will be tested in the next examples.

Example 5.2

As in the previous example the interpolation points will be uniformly distributed, cf. Figure 2. However, here we will test the point preconditioners proposed earlier in this paper. The tolerance ϵ is now reduced to 10^{-8}. The results obtained by using the explicit and implicit point preconditioner, combined with the conjugate gradient method for the normal equations, is reported in Tables 3 and 4, respectively. We observe that both preconditioners give a substantial improvement as compared to the results for simple diagonal operator in the previous example. For both preconditioners the condition numbers increase slowly with increasing number of interpolation points, and clearly the implicit preconditioner shows the best performance. Also obseve from Table 4 that the condition numbers are nearly independent of h, clearly indicating that the multigrid preconditioner removes the bad conditioning caused by a fine grid. These results indicates that impicit preconditioner is favorable as compared to the explicit preconditioner. The extra cost represented by the overhead needed for the implicit preconditioner is in fact rather small compared to total computational cost.

m	49=7×7		10=1×10		100=10×10		900=30×30		1600=40×40	
h	κ	# it	κ	# it	κ	# it	κ	# it	κ	# it
2^{-3}	8.1	58	7.9	54						
2^{-4}	9.3	70	5.9	44	15.1	128				
2^{-5}	10.9	84	6.5	46	19.0	142	139.5	1070		
2^{-6}	11.6	80	7.0	46	19.7	140	129.3	974	192.0	1552
2^{-7}	11.8	78	7.2	48	20.7	128	174.7	1062	289.2	1662

Table 3
The condition numbers and the required number of iterations with the conjugate gradient method for the normal system using the explicit point preconditioner.

m	49=7×7		10=1×10		100=10×10		900=30×30		1600=40×40	
h	κ	# it	κ	# it	κ	# it	κ	# it	κ	# it
2^{-3}	4.5	34	4.3	34						
2^{-4}	4.3	34	3.7	26	4.7	38				
2^{-5}	4.3	34	4.0	28	4.0	36	16.4	106		
2^{-6}	4.3	38	4.1	30	4.2	38	10.6	86	24.9	166
2^{-7}	4.0	34	4.0	30	4.2	38	10.1	78	19.0	140

Table 4
The condition numbers and the required number of iterations with conjugate gradient method for the normal system using the implicit point preconditioner.

m h	49=7×7	10=1×10	100=10×10	900=30×30	1600=40×40
2^{-3}	16	19			
2^{-4}	16	15	23		
2^{-5}	15	14	19	76	
2^{-6}	14	13	15	61	110
2^{-7}	13	13	15	54	92

Table 5
The number of iterations with the conjugate residual method using the implicit point preconditioner.

The corresponding iteration count for implicit preconditioner, but where the conjugate residual method is used as the iterative method instead of conjugate gradient for the normal equations, is presented in Table 5. We observe, as expected, that the conjugate residual method is faster, but the reduction is seldom more than 50%.

Example 5.3

In this example we test the two proposed point preconditioners on a different distribution of the interpolation points. All the points are localized in one corner of Ω, in the subsquare $\Omega_d = (0, d) \times (0, d)$. The distribution is uniformly in Ω_d, and is visualized in Figure 3. In all the experiments the tolerance $\epsilon = 10^{-8}$ is used. We only present estimates for the condition numbers. The results for the implicit and the explicit preconditioner are given in the same table.

In Table 6 all the interpolation points is located inside the square $(0, 2^{-2}) \times (0, 2^{-2})$. As the number of interpolation points m is increased we observe that the condition number estimates for the implicit preconditioner grows rather slowly with m, and this preconditioner is again clearly favorable to the explicit one.

In Table 7 the number of interpolation points has been fixed ($m = 15 = 3 \times 5$), while we vary the parameter d. We observe that the condition numbers are not very sensitive to this variation.

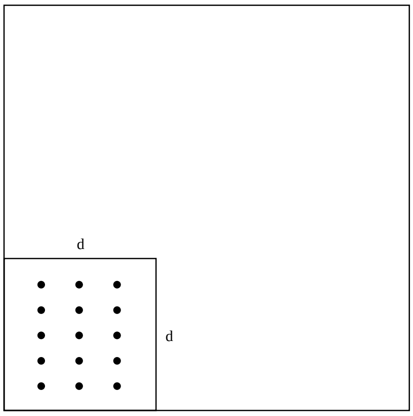

Figure 3. An example where the interpolation points are localized in a corner. We refer to this distribution as $\mathbf{D}_1$.

m	15=3×5		49=7×7		100=10×10		225=15×15	
h	Im	Ex	Im	Ex	Im	Ex	Im	Ex
2^{-4}	4.7	11.5						
2^{-5}	3.5	7.1	5.6	13.8				
2^{-6}	3.7	7.9	4.6	16.1	6.4	26.3	7.9	50.9
2^{-7}	4.0	8.6	4.0	18.0	4.7	33.3	6.9	57.1

Table 6
The condition numbers for the distribution $\mathbf{D}_1$. In this example $d = 2^{-2}$.

d	1		2^{-1}		2^{-2}		2^{-3}		2^{-4}	
h	Im	Ex	Im	Ex	Im	Ex	Im	Ex	Im	Ex
2^{-3}	3.6	4.6	4.7	10.7						
2^{-4}	3.9	5.1	3.4	6.5	4.7	11.5				
2^{-5}	4.2	5.7	3.6	7.3	3.5	7.1	4.7	11.6		
2^{-6}	4.2	5.7	4.0	8.0	3.7	7.9	3.5	7.2	4.7	11.6
2^{-7}	3.8	5.6	4.0	8.0	4.0	8.6	3.7	8.0	3.5	7.2

Table 7
The condition numbers for the distribution $\mathbf{D}_1$ with various d-values and $m = 15 = 3 \times 5$.

Example 5.4

We repeat the last experiment from the previous example above, by keeping the number of interpolation points fixed, but clustering the points as the parameter d decreases. We consider a point configuration where the interpolation points are clustered close to the boundary and at the center of Ω, cf. Figure 4. The results are given in Table 8. Again we observe that the condition numbers are nearly independent of d.

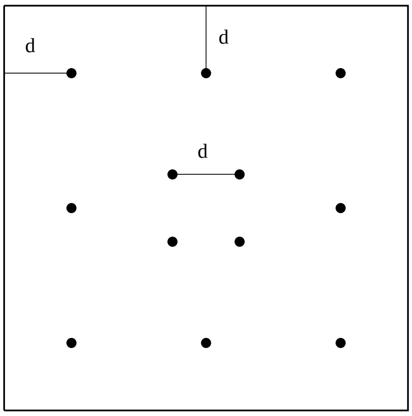

Figure 4. An example where the interpolation points is close to the boundary and to each other in the center of the square. We refer to this as distribution $\mathbf{D}_2$.

$h \backslash d$	2^{-2}		2^{-3}		2^{4}		2^{-5}	
	Im	Ex	Im	Ex	Im	Ex	Im	Ex
2^{-3}	2.7	3.3	4.3	3.6				
2^{-4}	3.0	4.0	4.4	4.6	4.8	4.2		
2^{-5}	3.3	4.4	4.7	5.5	4.7	5.1	5.6	4.0
2^{-6}	3.4	4.2	4.8	5.6	4.8	5.6	4.9	4.6
2^{-7}	3.1	4.1	4.3	5.6	4.6	5.7	4.8	5.3

Table 8

The condition numbers for the distribution $\mathbf{D}_2$ with various d-values. Here $m = 12$.

6. Conclusion

The scattered data interpolation problem is formulated as a differential–interpolation system of saddle point structure. We have verified experimentally that it is possible to precondition a finite element discretization of this system such that the condition numbers are practically independent of the mesh parameter h, independent of the distribution of the interpolation points, and almost independent of the number of interpolation points. We have also observed that the preconditioned conjugate residual method converges faster than the standard conjugate gradient method applied to the corresponding normal system.

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