Optimally Stable Multivariate Bases

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Abstract. We show that the tensor product B-spline basis and the triangular Bernstein basis are in some sense best conditioned among all nonnegative bases for the spaces of tensor product splines and multivariate polynomials, respectively. We also introduce some new condition numbers which are analogies of component-wise condition numbers for linear systems introduced by Skeel.

§1. Introduction

Conditioning of a problem measures the sensibility of the solution to perturbations of the data. If we focus on the problem of solving a linear system of equations $Ax = b$, the traditional condition number $\kappa(A) = \|A\| \, \|A^{-1}\|$ provides classical norm-wise perturbation results. Skeel performed a component-wise analysis of the problem, introducing a corresponding condition number which is often used, see [5].

In this paper we introduce some new condition numbers for function evaluation. These condition numbers are analogies of the Skeel condition number for linear systems and are useful when discussing optimal stability of nonnegative bases for finite dimensional function spaces. Loosely speaking a basis is optimally stable if it has the smallest condition number among all bases belonging to a certain class. In geometric modeling one is particularly interested in bases which are nonnegative in the domain of interest and we restrict our discussion on optimality to nonnegative bases.

A striking property of B-splines is the optimal stability of the basis. A similar property holds for polynomials and the Bernstein basis on $[0, 1]$, see [2,9]. In this paper we extend some of the univariate results on optimal stability to the multivariate case. In particular, we show that the tensor product B-spline basis and the triangular Bernstein basis for polynomials are optimally stable.

To decide whether a basis is well-conditioned or ill-conditioned one needs estimates of some condition number. For work on this for the B-spline basis and the Bernstein basis see [1,6,7,8,10]. We present a result showing that the new "component-wise" condition numbers for function evaluation can be as large as the norm-wise condition number, but it can also be much smaller.

Mathematical Methods for Curves and Surfaces II
Morten Dæhlen, Tom Lyche, Larry L. Schumaker (eds.), pp. 1–8.

ISBN 1-xxxxx-xxx-x.

§2. Condition Numbers for the Evaluation of Functions

Let $\mathcal{U}$ be a finite dimensional vector space of functions defined on $\Omega \subset \mathbb{R}^s$ and let $b = (b_0, \ldots, b_n)$ be a basis for $\mathcal{U}$. Given a function $f = \sum_{i=0}^n c_i b_i \in \mathcal{U}$ we are interested in measures for the sensitivity of $f(x)$ to perturbations in the coefficients $c = (c_j)_{j=0}^n$ of f. If $g = \sum_{i=0}^n (1+\delta_i) c_i b_i$ is related to f by a relative perturbation $\delta = (\delta_i)_{i=0}^n$ in c, then for any $x \in \Omega$

$$|f(x) - g(x)| = \Big|\sum_{i=0}^n \delta_i c_i b_i(x)\Big| \le ||\delta||_\infty \sum_{i=0}^n |c_i b_i(x)|.$$

The number

$$C_b(f,x) := \sum_{i=0}^n |c_i b_i(x)|, \tag{2.1}$$

acts as a condition number for the evaluation of f at the point x using the basis b. We can take the size of f into account and define related numbers by

$$\begin{aligned} \operatorname{cond}(b; f, x) &:= \frac{C_b(f,x)}{\|f\|_\infty} = \frac{\sum_{i=0}^n |c_i b_i(x)|}{\|\sum_{i=0}^n c_i b_i\|_\infty}, && (2.2)\\ \operatorname{cond}(b; f) &:= \sup_{x\in\Omega} \operatorname{cond}(b; f, x), && (2.3)\\ \operatorname{cond}(b) &:= \sup_{f\in\mathcal{U}} \operatorname{cond}(b; f). && (2.4) \end{aligned}$$

Since we use the norm of f in the definition of $\operatorname{cond}(b; f, x)$ we do not need to assume that $f(x) \ne 0$.

We observe that for $f, g \in \mathcal{U}$ as above and $x \in \Omega$

$$\begin{aligned} &|f(x) - g(x)| \le \varepsilon C_b(f,x) \\ &\frac{|f(x)-g(x)|}{||f||_\infty} \le \varepsilon \operatorname{cond}(b; f, x) \le \varepsilon \operatorname{cond}(b; f) \le \varepsilon \operatorname{cond}(b), \end{aligned} \tag{2.5}$$

where $\varepsilon = ||\delta||_\infty$. Thus the condition numbers can be used to measure the sensitivity of f to perturbations in the coefficients. A related condition number for roots of polynomials was considered by Farouki and Rajan in [3], see also [2,9].

There is another related ∞-norm condition number of the basis b. For all coefficients c we have

$$K_2^{-1} ||c||_\infty \le \Big|\Big|\sum_{i=0}^n c_i b_i\Big|\Big|_\infty \le K_1 ||c||_\infty, \tag{2.6}$$

where $K_1 = K_1(b)$ and $K_2 = K_2(b)$ are given by

$$K_1(b) := \sup_{c\ne 0} \frac{||\sum_{i=0}^n c_i b_i||_\infty}{||c||_\infty}, \quad \text{and} \quad K_2(b) := \sup_{c\ne 0} \frac{||c||_\infty}{||\sum_{i=0}^n c_i b_i||_\infty}.$$

The condition number $\kappa_\infty(b)$ is now defined as the product of K_1 and K_2

$$\kappa_\infty(b) := K_1(b)K_2(b). \tag{2.7}$$

This number was considered by de Boor in [1] for the B-spline basis and has been further studied in a series of papers, see [10] for an estimate which is very close to being best possible. We have a similar inequality to (2.5) for $\kappa_\infty(b)$. Indeed, from (2.6) we obtain

$$\frac{||f-g||_\infty}{||f||_\infty} \le \varepsilon\kappa_\infty(b),$$

where f, g, and ε are as above. As for the relation between $\mathrm{cond}(b)$ and $\kappa_\infty(b)$ it is easy to show that

$$\mathrm{cond}(b) \le \kappa_\infty(b) \tag{2.8}$$

whenever b is a basis of *blending* functions, *i.e.*, each b_i is nonnegative on Ω and $\sum_{i=0}^n b_i(x) = 1$ for all $x \in \Omega$. Indeed, $K_1(b) = 1$ for a basis of blending functions so that

$$\mathrm{cond}(b) = \sup_{c\neq 0} \frac{||\sum_{i=0}^n |c_i b_i|\ ||_\infty}{||f||_\infty} \le \sup_{c\neq 0} \frac{||c||_\infty}{||f||_\infty} = \kappa_\infty(b).$$

Let $e = (1,\ldots,1)^T$. We can shed some further light on the relation between the condition numbers introduced by recalling the following condition numbers for a nonsingular matrix A and a *vector* f

$$\begin{aligned}
\mathrm{Cond}(A,f) &:= \frac{||\,|A^{-1}|\,|A|\,|f|||_\infty}{||f||_\infty},\\
\mathrm{Cond}(A) &:= \sup_{f\neq 0}\mathrm{Cond}(A,f) = \mathrm{Cond}(A,e) = ||\,|A^{-1}|\,|A|\,||_\infty\\
\kappa(A) &:= ||A||_\infty\,||A^{-1}||_\infty.
\end{aligned}$$

Here the entries of $|A|$ are the absolute values of the corresponding entries in A. The numbers $\mathrm{Cond}(A,f)$ and $\mathrm{Cond}(A)$ where introduced by Skeel in 1979, see [5], and measures effects of perturbations of the data in linear systems $Af = c$. As an example we consider a perturbation in the right hand side c. Suppose this perturbation is of the form $\Delta\, c$ with $\Delta = \mathrm{diag}(\delta_0,\ldots,\delta_n)$ a diagonal matrix containing the relative perturbations $\delta = (\delta_0,\ldots,\delta_n)$. If $A(f+\delta f) = c + \Delta\, c$ we find

$$|\delta f| = |A^{-1}\Delta\, c| \le ||\delta||_\infty |A^{-1}|\cdot|c| \le ||\delta||_\infty |A^{-1}||A|\,|f|,$$

and so, if $||\delta||_\infty = \varepsilon$, we get the following analogue of (2.5)

$$\frac{||\delta f||_\infty}{||f||_\infty} \le \varepsilon\mathrm{Cond}(A,f) \le \varepsilon\mathrm{Cond}(A) \le \varepsilon\kappa(A).$$

We can observe the following two properties (see also Section 7.2 of [2]):

P1. Clearly $\mathrm{Cond}(A) \le \kappa_\infty(A)$ and it can be much smaller.

P2. In contrast to $\kappa(A)$, $\mathrm{Cond}(A)$ is invariant under row scaling: if D is a nonsingular diagonal matrix then

$$\mathrm{Cond}(DA) = \mathrm{Cond}(A).$$

These properties provide some of the reasons which explain why the Skeel condition number $\mathrm{Cond}(A)$ is more satisfying than the traditional condition number $\kappa(A)$.

As we have seen similar properties hold for the function condition numbers (2.2)-(2.4), and (2.7). By (2.8), the property P1 holds for bases of blending functions and P2 holds for $\mathrm{cond}(b)$. In fact, if we replace $b = (b_0, \ldots, b_n)$ by $\bar{b} = (k_0 b_0, \ldots, k_n b_n)$ ($k_i \in \mathbb{R}\ \forall\, i$) then $\mathrm{cond}(b) = \mathrm{cond}(\bar{b})$.

2.1. The Bernstein Basis

In this subsection we will compare $\mathrm{cond}(b; f, x)$ and $\kappa_\infty(b)$ for the univariate Bernstein basis

$$b_j(x) = \binom{n}{j} x^j (1-x)^{n-j}, \quad j = 0, \ldots, n,$$

and with

$$f(x) = T_n(2x-1)$$

the Chebyshev polynomial of degree n relative to the interval $[0,1]$. This function f is extremal for the sup $K_2(b)$ in (2.7) when b is the Bernstein basis, see [6,7]. Also, since b is a blending basis we have $K_1(b) = 1$. Since

$$T_n(x) = \sum_{j=0}^{n} (-1)^{n-j} \gamma_j b_j(x),$$

where $\gamma_0 = 1$ and

$$\gamma_j = \binom{2n-1}{2j-1} \Big/ \binom{n-1}{j-1}, \quad j = 1, \ldots, n$$

the largest coefficient γ_j is the middle one and

$$\kappa_\infty(b) = \frac{\gamma_{[n/2]}}{||f||_\infty} = \gamma_{[n/2]}.$$

As shown in [6] this gives the asymptotic estimate

$$\frac{n}{n+1} 2^{n-1/2} \le \kappa_\infty(b) \le \frac{n+1}{n} 2^{n-1/2}, \quad n \ge 1.$$

It is interesting to compare this with $\mathrm{cond}(b; f, x)$ for different x's. For $x = 1$ we find

$$\mathrm{cond}(b; f, x) = \gamma_n b_n(1) = 1, \quad n \ge 1$$

which is much smaller than $\kappa_\infty(b)$ for large n. On the other hand for $x = 1/2$ we obtain

$$\mathrm{cond}(b; f, x) = \sum_{j=0}^{n} \binom{n}{j} 2^{-n} \gamma_j = 2^{-n} \sum_{j=0}^{n} \binom{2n}{2j} = 2^{n-1}, \quad n \ge 1.$$

Thus $\mathrm{cond}(b; f, x)$ is essentially as large as $\kappa_\infty(b)$ for this choice of f and x.

§3. Some Auxilliary Lemmas on Optimal Stability.

Given a set $\mathcal{B}$ of bases of a vector space $\mathcal{U}$ of functions defined on Ω, we say that a basis $b \in \mathcal{B}$ is *optimally stable* for the evaluation of functions if there does not exist (up to permutation or scaling) a basis $u \in \mathcal{B}$ such that $\mathrm{cond}(u; f, x) \leq \mathrm{cond}(b; f, x)$ for each function $f \in \mathcal{U}$ evaluated at every value $x \in \Omega$. In the rest of this paper we consider the set $\mathcal{B}$ of bases of $\mathcal{U}$ formed by nonnegative functions.

Let u, v be two bases of $\mathcal{U}$. It obviously follows from the definitions that $\mathrm{cond}(u; f, x) \leq \mathrm{cond}(v; f, x)$ if and only if $C_u(f, x) \leq C_v(f, x)$. Thus, we could have defined optimal stability in terms of the condition number (2.1) instead of the condition number (2.2). Also one can easily check that if $\mathrm{cond}(u; f, x) \leq \mathrm{cond}(v; f, x)$ $\forall f \in \mathcal{U}$, $\forall x \in \Omega$ then $\mathrm{cond}(u) \leq \mathrm{cond}(v)$.

As in the previous section, for any $x \in \Omega \subseteq \mathbb{R}^s$, basis $u = (u_0, \ldots, u_n)$ and $f = \sum_{i=0}^n c_i u_i$, let us consider the relative condition number for the evaluation of f at x:

$$\mathrm{cond}(u; f, x) := \frac{\sum_{i=0}^n |c_i u_i(x)|}{\|\sum_{i=0}^n c_i u_i\|_\infty}.$$

The following result allows us to compare different bases of a space with respect to the previous condition number:

Lemma 3.1. *Let $\mathcal{U}$ be a finite dimensional vector space of functions defined on $\Omega \subset \mathbb{R}^s$. Let u, v be two bases of nonnegative functions of $\mathcal{U}$. Then*

$$\mathit{cond}(u; f, x) \leq \mathit{cond}(v; f, x), \quad \forall f \in \mathcal{U},\ \forall x \in \Omega \tag{3.1}$$

if and only if the matrix A such that $v = uA$ is nonnegative.

Proof: Let $A = (a_{ij})_{0 \leq i,j \leq n}$ be the matrix of change of basis such that $v = uA$. Suppose first A is nonnegative. If $f(x) = \sum_{j=0}^n c_j v_j(x)$ then we can write $f(x) = \sum_{i=0}^n (\sum_{j=0}^n c_j a_{ij}) u_i(x)$. Since A, u and v are nonnegative we find for every $x \in \Omega$

$$\sum_{j=0}^n |c_j v_j(x)| = \sum_{j=0}^n |c_j| v_j(x) = \sum_{i=0}^n (\sum_{j=0}^n |c_j| a_{ij}) u_i(x) \geq \sum_{i=0}^n \Big|\sum_{j=0}^n c_j a_{ij}\Big| u_i(x)$$

and therefore $\mathrm{cond}(u; f, x) \leq \mathrm{cond}(v; f, x)$.

Conversely, let us assume that (3.1) holds and assume that A is not nonnegative. Then $a_{pq} < 0$ for some p, q and the nonnegativity of the basis function u_p implies that there exists $x \in \Omega$ such that $u_p(x) > 0$. Since $v = uA$ we have

$$v_q(x) = a_{0q} u_0(x) + \cdots + a_{pq} u_p(x) + \cdots + a_{nq} u_n(x).$$

From the fact that $a_{pq} u_p(x) < 0$ and $v_q(x) \geq 0$, we deduce that there exists m such that $a_{mq} u_m(x) > 0$. Thus, taking into account that a_{pq} and a_{mq} are of opposite sign and the functions in u are nonnegative, we may deduce that

$$\begin{aligned} \mathrm{cond}(u; v_q, x) &= \frac{\sum_{i=0}^n |a_{iq}| u_i(x)}{\|v_q\|_\infty} \\ &> \frac{|\sum_{i=0}^n a_{iq} u_i(x)|}{\|v_q\|_\infty} = \frac{|v_q(x)|}{\|v_q\|_\infty} = \mathrm{cond}(v; v_q, x), \end{aligned}$$

which contradicts (3.1). $\square$

Lemma 3.2. *Let M be a nonsingular and nonnegative matrix such that the first and last nonzero entry of each column of M^{-1} is positive. Then $M = PD$, where D is a diagonal matrix with positive diagonal elements and P is a permutation matrix.*

Proof: Since the first nonzero element in each column of M^{-1} is positive it follows from Lemma 2.2 in [9] that $M^{-1} = LP_1$, where L is lower triangular and P_1 is a permutation matrix. Using that the last element in each column of M^{-1} is positive it can be shown that L is diagonal, see the last paragraph on page 1558 in [9]. But then $M = P_1^T L$ is of the required form. The diagonal elements of L must be positive since M is nonnegative and nonsingular. $\square$

Lemma 3.3. *Let u and b be nonnegative bases for a finite dimensional vector space $\mathcal{U}$ of functions defined on a set $\Omega \in \mathbb{R}^s$. Suppose that the change of basis matrix A defined by $b = uA$ is nonnegative and that the first and last element in each column of A^{-1} is positive. Then $u = b$ up to a permutation and positive scaling.*

Proof: By Lemma 3.2 it follows that $A = PD$ where P is a permutation matrix and D is a diagonal matrix with positive diagonal elements. But then $u = bA^{-1} = bD^{-1}P^T$ and the proof is complete. $\square$

We also need the following well known properties of the B-spline basis normalized to form a blending basis (see [11] and Lemma 2.1 of [9]).

Lemma 3.4. *Suppose $b = (N_{0,k}, \ldots, N_{n,k})$, where $N_{i,k}(t)$ is the ith B-spline of order k (degree $k-1$) on the knots $(\tau_i)_{i=0}^{n+k}$ with $\tau_{i+1} \geq \tau_i$ for all i and $\tau_{i+k} > \tau_i$ for $i = 0, \ldots, n$. Then*

$$\lim_{t \to \tau_j^+} \frac{N_{i,k}(t)}{N_{j,k}(t)} = 0, \quad \text{if } i > j, \tag{3.2}$$

and

$$\lim_{t \to \tau_{j+k}^-} \frac{N_{i,k}(t)}{N_{j,k}(t)} = 0, \quad \text{if } i < j. \tag{3.3}$$

§4. Optimal Stability of Tensor Product B-splines.

Consider now s-variate tensor product polynomial splines defined on a rectangle

$$\Omega = [a_1, d_1] \times [a_1, d_1] \times \cdots \times [a_s, d_s].$$

We let for $x = (x_1, \ldots, x_s) \in \Omega$, $\alpha = (\alpha_1, \ldots, \alpha_s)$, and $0 \leq \alpha_r \leq n_r$

$$b_\alpha(x) = N_{\alpha_1,k_1,\tau^1}(x_1) \cdots N_{\alpha_s,k_s,\tau^s}(x_s) \tag{4.1}$$

be the tensor product B-splines, where $(N_{i,k_r,\tau^r})_{i=0}^{n_r}$ are the usual univariate B-splines on knots $(\tau_i^r)_{i=0}^{n_r+k_r}$. We have the corresponding space

$$\mathcal{U} = \{ \sum_{0\le\alpha\le n} c_\alpha b_\alpha : c_\alpha \in \mathbb{R}\} \tag{4.2}$$

of tensor product polynomial splines on Ω. Here $n = (n_1, \ldots, n_s)$ and we order the elements of $b = (b_\alpha)$ in lexicographical order.

The following result proves that the tensor product B-spline basis is optimally stable.

Theorem 4.1. *Let $b = (b_\alpha)$ be the tensor product B-spline basis (4.1) for the corresponding space $\mathcal{U}$ given by (4.2). If $u = (u_\alpha)$ is a basis of nonnegative functions for $\mathcal{U}$ such that*

$$cond(u; f, x) \le cond(b; f, x) \tag{4.3}$$

for each function $f \in \mathcal{U}$ evaluated at every value $x \in \Omega$, then $u = b$ up to permutation and positive scaling.

Proof: By Lemma 3.1 there exists a nonnegative matrix A such that

$$b = uA$$

and by Lemma 3.3 it is enough to show that the first and last element in each column of A^{-1} is positive. Fix a column β of A^{-1} and let $(c_\alpha)_{0\le\alpha\le n}$ be the elements of A^{-1} in this column so that

$$u_\beta(x) = \sum_\alpha c_\alpha b_\alpha(x). \tag{4.4}$$

Let $\alpha = (l_1, \ldots, l_s)$ be the index of the first nonzero c_α in (4.4). We prove that it is positive by induction on s. If $s = 1$, the function u_β of (4.4) can be written as

$$u_\beta(x_1) = c_{l_1} N_{l_1,k_1}(x_1) + \sum_{\alpha_1 > l_1} c_{\alpha_1} N_{\alpha_1,k_1}(x_1). \tag{4.5}$$

Using the nonnegativity of u_β and N_{l_1,k_1} joint with (4.5) and (3.2) we deduce that

$$0 \le \lim_{x_1 \to (\tau_{l_1}^1)^+} \frac{u_\beta(x_1)}{N_{l_1,k_1}(x_1)} = c_{l_1}.$$

Assume that the result holds for $s-1$ and let us prove it for s. The function u_β of (4.4) can be written as

$$u_\beta(x) = \sum_{\alpha_1=l_1}^{n_1} N_{\alpha_1,k_1}(x_1) \left(\sum_{\alpha_2=0}^{n_2} \cdots \sum_{\alpha_s=0}^{n_s} c_\alpha N_{\alpha_2,k_2}(x_2) \cdots N_{\alpha_s,k_s}(x_s) \right). \tag{4.6}$$

Using the nonnegativity of u_β and N_{l_1,k_1} we deduce that

$$0 \le \lim_{x_1 \to (\tau^{(1)}_{l_1})^+} \frac{u_\beta(x_1)}{N_{l_1,k_1}(x_1)} \tag{4.7}$$

By (4.6), (4.7) and (3.2) we conclude that

$$\sum_{\alpha_2=0}^{n_2} \cdots \sum_{\alpha_s=0}^{n_s} c_{(l_1,\alpha_2,\ldots,\alpha_s)} N_{\alpha_2,k_2}(x_2) \cdots N_{\alpha_s,k_s}(x_s) \ge 0. \tag{4.8}$$

Since the function in (4.8) depends on $s-1$ variables, by our induction hypothesis its first nonzero coefficient is positive. This coefficient is $c_{(l_1,l_2,\ldots,l_s)}$ and coincides with the the first nonzero coefficient in (4.4), so that the result follows.

Let c_α, $\alpha_r = m_r$ $\forall\, r = 1,\ldots,s$, be the last nonzero coefficient in (4.4). Applying a reasoning analogous to the proof of the positivity of the first nonzero coefficient in (4.4), but using (3.3) instead of (3.2) and taking limits when $x_s \to (\tau^s_{m_s+k_s})^-$, we may deduce that the last nonzero coefficient in (4.4) must be also positive, that is, the last nonzero entry of each column of A^{-1} is positive. $\square$

For an s-tuple $n = (n_1,\ldots,n_s)$ consider next the tensor product space

$$\mathcal{U} = \{ \sum_{0 \le \alpha \le n} c_\alpha x^\alpha : c_\alpha \in \mathbb{R} \}$$

of polynomials in s variables $x = (x_1,\ldots,x_s)$ of degree $\le n_r$ in the variable x_r for $r = 1,\ldots,s$. A basis for this space is the Bernstein basis

$$b_{\alpha_1,\ldots,\alpha_s}(x_1,\ldots,x_s) = B^{n_1}_{\alpha_1}(x_1)\cdots B^{n_s}_{\alpha_s}(x_s), \quad 0 \le \alpha_r \le n_r,\ r = 1,\ldots,s, \tag{4.9}$$

where

$$B^{n_r}_{\alpha_r}(x_r) := \binom{n_r}{\alpha_r}(x_r)^{\alpha_r}(1-x_r)^{n_r-\alpha_r}, \quad \alpha_r = 0,1,\ldots,n_r, \quad x_r \in [0,1]$$

is the univariate Bernstein basis function of degree n_r for $r = 1,\ldots,s$.

Taking into account that the Bernstein basis is a particular B-spline basis we can derive the following consequence of Theorem 4.1:

Corollary 4.2. *Let b be the tensor product Bernstein basis for the corresponding space $\mathcal{U}$. If u is a basis of nonnegative functions for $\mathcal{U}$ such that*

$$cond(u; f, x) \le cond(b; f, x)$$

for each function $f \in \mathcal{V}$ evaluated at every value $x \in [0,1]^s$ then $u = b$ up to permutation and positive scaling.

§5. Optimal Stability of the Triangular Bernstein Basis.

Let us now introduce the triangular Bernstein basis. Let $i = (i_1, \ldots, i_s)$ denote a multi-index with $i_1, \ldots, i_s$ in $\mathbb{Z}_+$. We denote by $|i|$ the sum of the coefficients $i_1 + \cdots + i_s$. Let $i! = i_1! \cdots i_s!$ and $i^\alpha = i_1^{\alpha_1}, \ldots, i_s^{\alpha_s}$. For the vector space of polynomials of total degree at most n in s variables $x = (x_1, \ldots, x_s)$

$$\mathcal{U} = \{ \sum_{|i| \le n} c_i x^i | c_i \in \mathbb{R} \},$$

let us consider the triangular Bernstein basis

$$\{ \frac{n!}{\alpha!} \lambda^\alpha \}_{|\alpha|=n}, \tag{5.1}$$

where $(\lambda_1, \ldots, \lambda_{s+1})$ denotes the barycentric coordinate with respect to a non-degenerate simplex $\Omega =< v_1, \ldots, v_{s+1} >$ in $\mathbb{R}^s$, that is, the $(s+1)$-tuple λ corresponding to x is uniquely given by

$$\sum_{j=1}^{s+1} \lambda_j v_j = x, \quad \sum_{j=1}^{s+1} \lambda_j = 1.$$

Note that $\lambda^\alpha = \lambda_1^{\alpha_1} \cdots \lambda_s^{\alpha_s} (1 - \lambda_1 - \cdots - \lambda_s)^{n - \alpha_1 - \cdots - \alpha_s}$, and that $\lambda \ge 0$ when $x \in \Omega$ so that the basis is nonnegative on Ω. It was shown in [4] that the bivariate Bernstein basis is better conditioned than the corresponding power basis. We now show that the triangular Bernstein basis (5.1) is optimally stable.

Theorem 5.1. *Let b be the Bernstein basis (5.1) for the space $\mathcal{U}$ of multivariate polynomials of total degree $\le n$. If u is another basis for $\mathcal{U}$ of functions which are nonnegative on Ω and such that*

$$cond(u; f, x) \le cond(b; f, x)$$

for each function $f \in \mathcal{U}$ evaluated at every value $x \in \Omega$ then $u = b$ up to permutation and positive scaling.

Proof: Let us introduce the change of variables (see p. 143 of [7])

$$x \to y = (y_1, \ldots, y_s)$$

given by

$$\begin{aligned}
\lambda_1 &= y_1 && = y_1 \\
\lambda_2 &= y_2(1 - y_1) && = y_2(1 - \lambda_1) \\
\lambda_3 &= y_3(1 - y_2)(1 - y_1) && = y_3(1 - \lambda_1 - \lambda_2) \\
&\vdots && \vdots \\
\lambda_s &= y_s(1 - y_{s-1}) \cdots (1 - y_1) && = y_s(1 - \lambda_1 - \lambda_2 - \cdots - \lambda_{s-1}).
\end{aligned}$$

This transformation maps Ω onto the s-dimensional cube $[0,1]^s$. Since

$$\lambda_{s+1} = 1 - \lambda_1 - \cdots - \lambda_s = (1-y_s)(1-y_{s-1})\cdots(1-y_1)$$

we derive

$$\lambda^\alpha = y_1^{\alpha_1}\cdots y_s^{\alpha_s}(1-y_1)^{n-\alpha_1}\cdots(1-y_s)^{n-\alpha_1-\cdots-\alpha_s}.$$

Taking into account that

$$\frac{n!}{\alpha!} = \binom{n}{\alpha_1}\binom{n-\alpha_1}{\alpha_2}\binom{n-\alpha_1-\alpha_2}{\alpha_3}\cdots\binom{n-\alpha_1-\cdots-\alpha_{s-1}}{\alpha_s}$$

we conclude that

$$\frac{n!}{\alpha!}\lambda^\alpha = B^n_{\alpha_1}(y_1)B^{n-\alpha_1}_{\alpha_2}(y_2)\cdots B^{n-\alpha_1-\cdots-\alpha_{s-1}}_{\alpha_s}(y_s). \tag{5.2}$$

Thus any function $f = \sum_{|\alpha|=n} c_\alpha \frac{n!}{\alpha!}\lambda^\alpha$ in $\mathcal{U}$ can be written in tensor product manner as

$$\sum_{\alpha_1=0}^{n} B^n_{\alpha_1}(y_1)\left(\sum_{\alpha_2=0}^{n-\alpha_1} B^{n-\alpha_1}_{\alpha_2}(y_2)\cdots\sum_{\alpha_s=0}^{n-\alpha_1-\cdots-\alpha_{s-1}} c_\alpha B^{n-\alpha_1-\cdots-\alpha_{s-1}}_{\alpha_s}(y_s)\right). \tag{5.3}$$

The problem now reduces to prove the optimal stability of the basis (5.1) using the new expressions (5.2) with respect to the new variables. We can reason similarly as in the proof of Theorem 4.1, using again induction on the number of variables s appearing in the expression (5.3) of the functions in $\mathcal{U}$. Now, the expression of $u_\beta(x)$ in (4.6) should be replaced by

$$\sum_{\alpha_1=l_1}^{n} B^n_{\alpha_1}(y_1)\left(\sum_{\alpha_2=0}^{n-\alpha_1} B^{n-\alpha_1}_{\alpha_2}(y_2)\cdots\sum_{\alpha_s=0}^{n-\alpha_1-\cdots-\alpha_{s-1}} c_\alpha B^{n-\alpha_1-\cdots-\alpha_{s-1}}_{\alpha_s}(y_s)\right) \tag{5.4}$$

and formula (4.8) should be replaced by

$$\sum_{\alpha_2=0}^{n-l_1}\cdots\sum_{\alpha_s=0}^{n-l_1-\alpha_2-\cdots-\alpha_{s-1}} c_{l_1,\alpha_2,\ldots,\alpha_s} B^{n-l_1}_{\alpha_2}(y_2)\cdots B^{n-l_1-\alpha_2-\cdots-\alpha_{s-1}}_{\alpha-s}(y_s) \geq 0. \tag{5.5}$$

The functions in (5.5) have a similar form to those of (5.3) but using $s-1$ variables instead of s variables. So the induction hypothesis can be applied analogously as in the proof of Theorem 4.1. $\square$

Acknowledgments. *The second author was partially supported by the Research Grant BFM 2000-1253.*

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