A Hermite Subdivision Scheme for the Evaluation of the Powell-Sabin 12-Split Element

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Abstract. It is observed that the Powell-Sabin 12-split triangle is refinable since the same split of the 4 similar subtriangles of a triangle contains the lines of split of the original triangle. This property of the split is the key to the existence of a subdivision scheme, for the evaluation of the C^1 quadratic spline on the split which interpolates function and gradient values at the 3 vertices of the triangle, and normal derivatives at the midpoints of the edges. Explicit formulae for the Hermite subdivision step are given. For rendering the interpolant it is suggested to use the triangulation and the function values at the vertices obtained after a small number of subdivision iterations, and to use the known values of the gradient at the vertices to obtain the normals to the surface at the vertices of the triangulation. The shading of the 3D triangulation can then be done by Gouraud shading. It is further suggested to perturb the C^1-Hermite subdivision scheme which evaluates the above interpolant on the Powell-Sabin 12-split triangle, to obtain other C^1 schemes with a shape parameter.

§1. Introduction

For bivariate smooth spline spaces of low degree on triangulations it is difficult to construct basis functions with local support. One approach which leads to good results is the splitting of each triangle into subtriangles according to the same rule of splitting. Among the known splits is the Powell-Sabin 12- split (PS-12 split). Here each triangle is divided into 12 subtriangles by connecting each vertex of the triangle to the midpoint of the opposite edge and connecting the midpoints, see Fig. 1. On this split there is a unique quadratic C^1-spline interpolant to function values and gradients at the vertices and cross derivatives at the midpoints of the three edges. ([9]). This interpolant is called the *PS-12 split element*.

Due to the large number of subtriangles in this split it is hard to compute these elements, (but see [1]). Yet this split has the following advantage ([8,7]):

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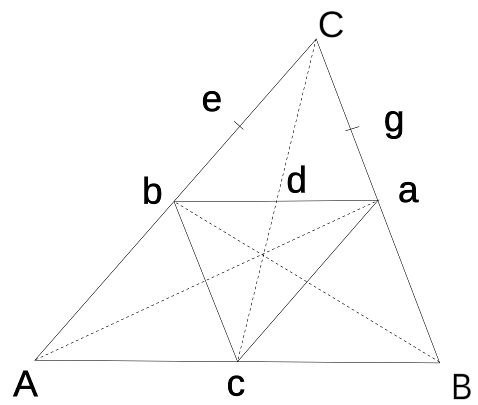

Fig. 1. A triangle divided into 12 subtriangles by the PS-12 split.

Suppose we subdivide the original triangle into 4 similar triangles (called refined triangles) by connecting midpoints of edges. Then using the PS-12 split on each of the refined triangles we obtain a spline space of C^1 piecewise quadratic polynomials which contains the spline space of C^1 piecewise quadratic polynomials defined on the PS-12 split of the original triangle as a subspace. This property is called *refinability* and is important for multiresolution analysis. Other well-known splits do not have this property. This refinability property allows us to compute the PS-12 split element in a simple way, which ignores the split and generates function values and gradients on a dense set of points. By repeatedly subdividing the refined triangles in this way we obtain the values and gradients of the PS-12 split element on the vertices of the refined triangulation. We can stop this process after a few iterations of the subdivision (4-6 itetations) and display the piecewise linear surface on the current refined triangulation, using the function values. To improve the rendering of the surface one can use the gradients for evaluating the normals at the vertices of the current triangulation, and perform Gouraud shading of the piecewise linear surface. (Our figures of the subdivision surfaces were obtained without the Gouraud shading, which generates the same qaulity of rendered surfaces after 2-3 iterations).

In this paper we show how to compute the PS-12 split element by a Hermite subdivision scheme on triangles. A different C^1-Hermite subdivision scheme was designed in [3], with a limit surface which is not a spline. Univariate Hermite subdivision schemes were studied in [2,5,6]. Special cases of these schemes lead to univariate splines.

By perturbing the formula we obtain a one-parameter family of new C^1-Hermite subdivision schemes with the perturbation parameter acting as a *tension parameter*. The resulting surfaces are no longer spline surfaces.

§2. Computing the PS-12 Split Element by Subdivision.

Given function values and gradients at the vertices A, B, C of a triangle T, and cross derivatives at midpoints a, b, c of edges opposite A, B, C (see, Fig. 1), a unique PS-12 split element which interpolates this data is determined ([9]).

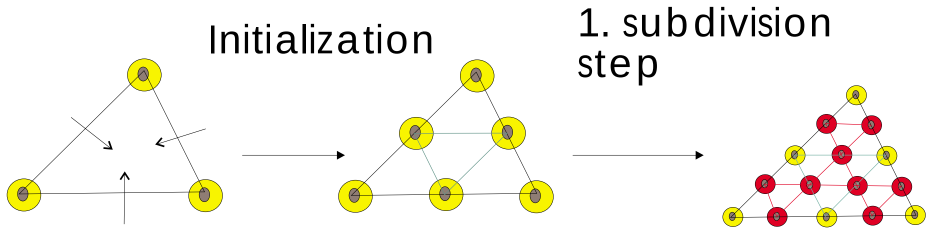

Fig. 2. Subdividing the PS-12 split element. A circle around a vertex means that both the function value and the gradient are known at that vertex.

2.1. Initialization.

The first step in the computation of such an element involves the computation of its value and gradient at the midpoints a, b, c of the triangle T, (see, Fig. 1). Here we use the formula

$$f_b = (f_A + f_C)/2 - (\nabla f_A - \nabla f_C) \cdot (A - C)/8$$

for the function value at the midpoint b of AC. For the gradient we first compute the directional derivative in the direction AC at b

$$(A - C) \cdot \nabla f_b = 2(f_A - f_C) - (\nabla f_A + \nabla f_C) \cdot (A - C)/2.$$

Combining this value with the given value of the cross-derivative at b, we can calculate ∇f_b. For the other midpoints we use similar formulae.

These formulae are obtained from the observation that along each side of T the PS-12 split element is a piecewise quadratic C^1-spline with a knot at the midpoint.

2.2. The General Subdivision Step.

For the first subdivision step (see Fig. 1) we use the following formulae:

$$\begin{aligned}
f_e &= (f_b + f_C)/2 - (\nabla f_b - \nabla f_C) \cdot (b - C)/8 \\
f_g &= (f_a + f_C)/2 - (\nabla f_a - \nabla f_C) \cdot (a - C)/8 \\
f_d &= (f_b + f_a)/2 - (\nabla f_b - \nabla f_a) \cdot (b - a)/8 \\
\nabla f_e &= (\nabla f_b + \nabla f_C)/2 \\
\nabla f_g &= (\nabla f_a + \nabla f_C)/2 \\
(a - b) \cdot \nabla f_d &= 2(f_a - f_b) - (\nabla f_a + \nabla f_b) \cdot (a - b)/2 \\
(C - d) \cdot \nabla f_d &= 2(f_C - f_d) - \nabla f_C \cdot (C - d).
\end{aligned} \tag{1}$$

¿From the last two values we can solve for ∇f_d. Similar formulae are used for the two other corner triangles Abc and Bca and we obtain the values and gradients at locations shown to the right in Fig. 2. This process can now be continued for as many levels of refinement as desired.

Fig. 3 displays a PS-12 split element obtained from random initial data. The implementation was done using Mathematica.

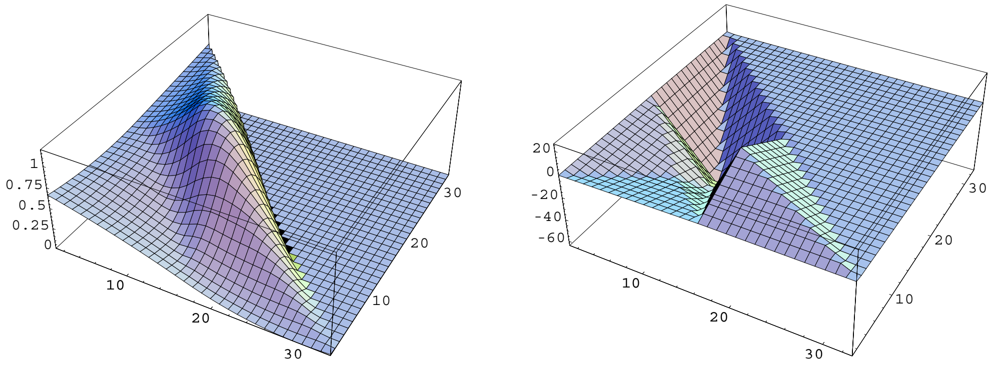

Fig. 3. A PS-12 split element (left) constructed from random data using 5 levels of subdivision. The partial derivative in the x-direction (right) .

2.3. Computation of Interpolants on a Given Triangulation.

The above procedure can be extended to any initial triangulation Δ with data consisting of function values and gradient values at the vertices, and cross derivatives at midpoints of edges. The data determines a unique interpolant f in the space $S_2^1(\Delta_{PS})$, where Δ_{PS} is the triangulation obtained from Δ by splitting each triangle of Δ into 12 subtriangles according to the PS-12 split. Here $S_d^r(\Delta)$ denotes the spline space of C^r-piecewise polynomials of degree $\leq d$ on the triangulation Δ.

With the subdivision procedure one can get values and gradients of the interpolant f at the vertices of a refined triangulation of the initial one, for any level of refinement.

§3. Tension Parameters.

Since the formulae (1) generate a C^1-surface by subdivision, small changes in parameters will also generate a C^1-surface. This follows from the observation that a subdivision scheme which depends on a parameter and generates a C^1-surface, will have this property in a neighborhood of that parameter value (see [4]).

Suppose in (1) we keep the formulae for the gradient values, but change the formulae for function values to

$$\begin{aligned} f_e &= (f_b + f_C)/2 - (\nabla f_b - \nabla f_C) \cdot (b - C)\alpha \\ f_g &= (f_a + f_C)/2 - (\nabla f_a - \nabla f_C) \cdot (a - C)\alpha \\ f_d &= (f_b + f_a)/2 - (\nabla f_b - \nabla f_a) \cdot (b - a)\alpha. \end{aligned} \tag{2}$$

Then for $\alpha = 1/8$ we get a C^1-surface. Therefore, for α in a neighborhood of $1/8$ we still get a C^1-surface.

We observe that for $\alpha = 0$ we obtain a piecewise linear interpolant to the data of function values at the vertices , see Fig. 4. Therefore, we expect to get a "tighter" surface in the left neighborhood of 1/8 and a "looser" surface in the right neighborhood of 1/8. Figs. 5,6 are in agreement with this expectation. This can be seen by comparing the range of function values of the partial derivative in the x-direction.

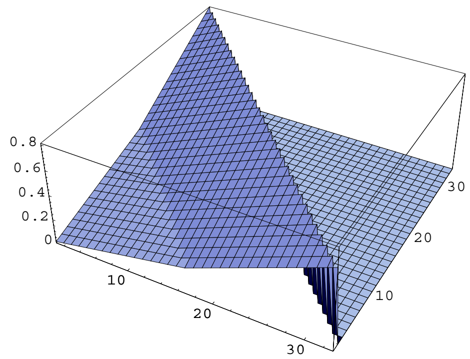

Fig. 4. Subdivision surface with tension parameter $\alpha = 0$ using the same initial data as in Fig. 3.

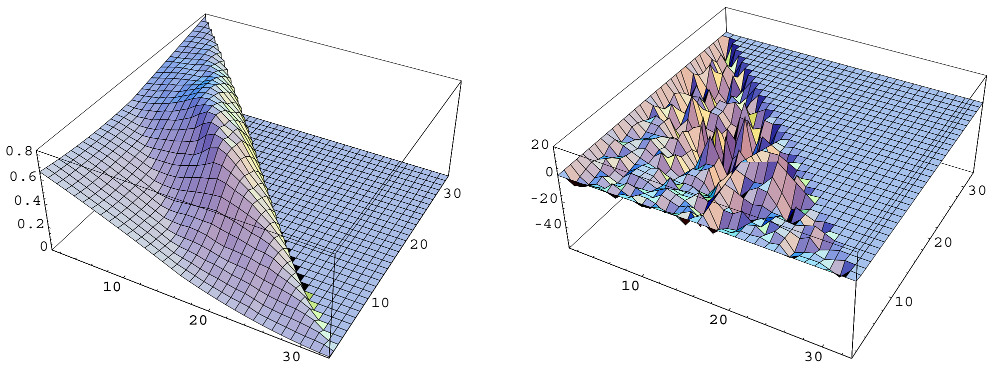

Fig. 5. Subdivision surface (left) with tension parameter $\alpha = 1/16$ using the same initial data as in Fig. 3. The partial derivative in the x-direction (right).

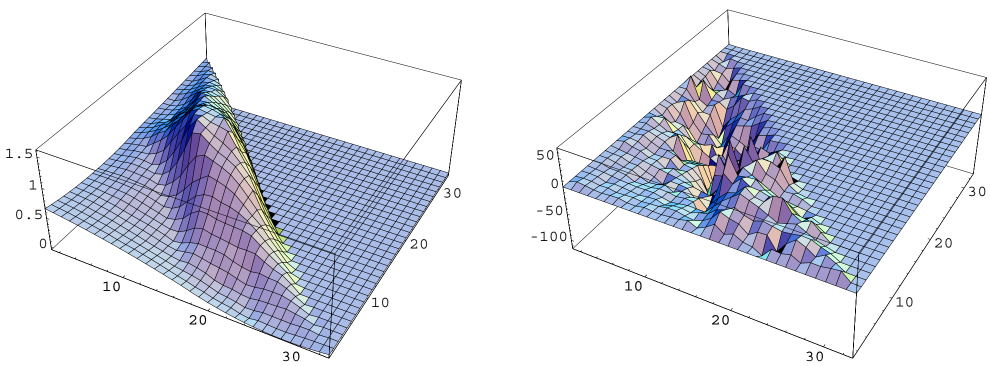

Fig. 6. Subdivision surface (right) with tension parameter $\alpha = 3/16$ using the same initial data as in Fig. 3. The partial derivative in the x-direction (right).

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References

1. Chui, C. K., and T.-X. He, Bivariate C^1 quadratic finite elements and vertex splines, Math. Comp. **54** (1990),169–187.
2. Merrien, J.-L., A family of Hermite interpolants by bisection algorithms, Numer. Algorithms **2** (1992),187–200.
3. Merrien, J.-L., Dyadic Hermite interpolants on a triangulation, Numer. Algorithms **7** (1994),391–410.
4. N. Dyn Subdivision schemes in Computer-Aided Geometric Design. In "Advances in Numerical Analysis - Volume II, Wavelets, Subdivision Algorithms and Radial Basis Functions", W. Light (ed.), Clarendon Press, Oxford, 36-104 (1992).
5. Dyn, N., and D. Levin, Analysis of Hermite-type subdivision schemes, in *Approximation Theory VIII, Vol. 2: Wavelets*, Charles K. Chui and Larry L. Schumaker (eds.), World Scientific Publishing Co., Inc., Singapore, 1995, 117-124.
6. Dyn, N., and D. Levin, Analysis of Hermite-interpolatory subdivision schemes. To appear in "Spline Functions and the Theory of Wavelets" S. Dubuc (ed.), AMS series CRM Proceedings and Lecture Notes, (1998).
7. Dæhlen, M., T. Lyche, K. Mørken, R. Schneider, and H-P Seidel, Multiresolution analysis based on quadratic Hermite interpolation, Part 2: Piecewise polynomial surfaces over triangles, preprint.
8. Oswald, P., Hierarchical conforming finite element methods for the biharmonic equation, SIAM J. Numer. Anal.**29**(1992),1610–1625.
9. Powell, M. J. D., and M. A. Sabin, Piecewise quadratic approximation on triangles, ACM Trans. Math. Software **3** (1977), 316-325.

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