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CONSTRUCTIVE APPROXIMATION

Addendum to "On the L_1-Condition Number of the Univariate Bernstein Basis"

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Abstract. The paper mentioned above is a contribution of the authors to Volume 18 (2002) of this journal, see [1]. Recently, J. Domsta pointed out to us that the proof of Theorem 4.2 in that paper contains an error. The purpose of this addendum is to present a correct argument for it.

1. Introduction

The 1-norm condition number of the Bernstein basis of degree n can be defined by

$$\kappa_{n,1} := \sup_{(c_j)\neq 0} \frac{\|\sum_{j=0}^{n} c_j B_j^n\|_1}{\sum_{j=0}^{n} |c_j|} \sup_{(c_j)\neq 0} \frac{\sum_{j=0}^{n} |c_j|}{\|\sum_{j=0}^{n} c_j B_j^n\|_1}, \tag{1.1}$$

where

$$B_j^n(x) := \binom{n}{j}\left(\frac{1+x}{2}\right)^j\left(\frac{1-x}{2}\right)^{n-j}, \qquad j = 0, \ldots, n, \tag{1.2}$$

is the Bernstein basis for polynomials of degree n relative to the interval $[-1, 1]$ and $\|f\|_1 := \int_{-1}^{1} |f(x)|\, dx$. In Section 4 of [1] we tried to show that an extremal solution f of the problem

$$\sup_{\substack{g\in\Pi_n\\ g\neq 0}} \frac{\|\mathbf{C}_n(g)\|_1}{\|g\|_1}, \tag{1.3}$$

where

$$\|\mathbf{C}_n(g)\|_1 := \sum_{j=1}^{n} |c_j|, \qquad g = \sum_{j=0}^{n} c_j B_j^n,$$

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or, equivalently, of the inf problem

$$(1.4)\qquad I_n(f) = \inf_{\substack{g\in\Pi_n\\ g\neq 0}} I_n(g), \qquad \text{where} \quad I_n(g) = \frac{\|g\|_1}{\|\mathbf{C}_n(g)\|_1},$$

must belong to the class

$$(1.5)\qquad \Pi_n^A := \left\{ \sum_{j=0}^{n} d_j B_j^n : d_{j-1}d_j < 0,\ j = 1, \dots, n \right\}$$

of polynomials of degree $\leq n$ with alternating coefficients. In this respect we stated the following slightly stronger result (Theorem 4.2 of [1]).

Theorem 1.1. *An extremal solution of* (1.3) *has n distinct roots in* $(-1, 1)$.

For its proof the following lemma (Lemma 4.1 in [1]) was used.

Lemma 1.2. *Suppose* $f = \sum_{j=0}^{n} d_j B_j^n$ *has precisely* $m \leq n$ *sign changes in* $(-1, 1)$. *Then there are integers* $i_0, \dots, i_m$ *with* $0 \leq i_0 < \cdots < i_m \leq n$ *such that*

$$(1.6)\qquad \begin{aligned} d_{i_{k-1}} d_{i_k} &< 0 \quad \text{for} \quad k = 1, \dots, m, \\ d_j d_{i_0} &\geq 0 \quad \text{for} \quad j = 0, \dots, i_0. \end{aligned}$$

Moreover, for any integers $j_0, \dots, j_m$ *with* $0 \leq j_0 < \cdots < j_m \leq n$ *there is a unique polynomial* $g = \sum_{j=0}^{n} c_j B_j^n$ *with the following properties*:

$$(1.7)\qquad \begin{aligned} &\text{(i)} \quad g = \sum_{k=0}^{m} c_{j_k} B_{j_k}^n \quad \text{so that} \quad c_j = 0 \quad \text{for} \quad j \notin \{j_0, \dots, j_m\}, \\ &\text{(ii)} \quad f(x)g(x) \geq 0 \quad \text{for} \quad x \in [-1, 1], \\ &\text{(iii)} \quad d_{i_k} c_{j_k} > 0 \quad \text{for} \quad k = 0, \dots, m, \\ &\text{(iv)} \quad \sum_{k=0}^{n} |c_k| = 1. \end{aligned}$$

We denote this polynomial by $g = R(f; i_0, \dots, i_m; j_0, \dots, j_m)$.

2. Corrected Proofs

In Lemma 1.2 it is tacitly assumed that $m \geq 1$ which was overlooked in [1]. Therefore, we have still to settle this before we can apply it. This is the content of

Lemma 2.1. *The extremal function for the second supremum in* (1.1) *must have at least one sign change.*

Proof. Recall that

$$B_j^n(x) \geq 0, \quad x \in [-1, 1], \quad \sum_{j=0}^{n} B_j^n(x) = 1, \quad x \in \mathbb{R}, \quad \text{and} \quad \int_{-1}^{1} B_j^n(x)\, dx = \frac{2}{n+1}.$$

Suppose $f = \sum_{j=0}^{n} d_j B_j^n$ is an extremal for the second sup in (1.1) or, equivalently, an extremal for the inf problem (1.4). We first show that the d_j's must change sign and use this to show that f also must have a sign change in $(-1, 1)$.

Suppose all the d_j's have the same sign. Then

$$I_n(f) = \frac{\int_{-1}^{1} |\sum_{j=0}^{n} d_j B_j^n(x)|\, dx}{\sum_{j=0}^{n} |d_j|} = \frac{\sum_{j=0}^{n} |d_j| \int_{-1}^{1} B_j^n(x)\, dx}{\sum_{j=0}^{n} |d_j|} = \frac{2}{n+1}.$$

But this value cannot be the infimum since for the degree n Chebyshev polynomial U_n of the second kind we have shown in Theorem 2.1 in [1] that

$$I_n(U_n) \le \frac{2^{1-n}}{n+1}\sqrt{\frac{n+2}{\pi}}.$$

We conclude that the d_j's must change sign at least once. Suppose next that the extremal f does not change sign in $(-1, 1)$ and is nonnegative. For $\delta > 0$ we define

$$Z_\delta := \{x \in (-1, 1) : f(x) \le \delta\}$$

and let $Z_\delta^c := [-1, 1]\backslash Z_\delta$ be the complement of Z_δ. Pick j_0 such that $d_{j_0} < 0$ and let $f_\delta := f - \delta B_{j_0}^n$. We will show that $I_n(f_\delta) < I_n(f)$ for $\delta > 0$ sufficiently small, a contradiction to the extremality of f.

Now since $f(x) > \delta$ on Z_δ^c and $0 \le B_{j_0}(x) \le 1$ on $[-1, 1]$ we find

$$\begin{aligned}
\|f_\delta\|_1 &= \int_{Z_\delta^c} |f(x) - \delta B_{j_0}^n(x)|\, dx + \int_{Z_\delta} |f(x) - \delta B_{j_0}^n(x)|\, dx \\
&\le \int_{Z_\delta^c} |f(x)|\, dx - \delta \int_{Z_\delta^c} B_{j_0}^n(x)\, dx + \int_{Z_\delta} |f(x)|\, dx + \delta \int_{Z_\delta} B_{j_0}^n(x)\, dx \\
&= \int_{-1}^{1} |f(x)|\, dx - \delta \int_{-1}^{1} B_{j_0}^n(x)\, dx + 2\delta \int_{Z_\delta} B_{j_0}^n(x)\, dx \\
&\le \|f\|_1 - 2\delta\left(\frac{1}{n+1} - \mu(Z_\delta)\right),
\end{aligned}$$

where $\mu(Z_\delta)$ is the measure of the set Z_δ. Since the polynomial f vanishes only at a finite number of isolated points we can choose $\delta > 0$ so small that $\mu(Z_\delta) < 1/(n+1)$, and we have shown that $\|f_\delta\|_1 < \|f\|_1$. ■

Corrected Proof of Theorem 1.1. Suppose there is an extremal polynomial

$$f = \sum_{j=0}^{n} d_j B_j^n, \qquad \text{with} \qquad \sum_{j=0}^{n} |d_j| = 1,$$

which has $m < n$ sign changes in $(-1, 1)$ and let the integers $i_0, \ldots, i_m$ be given by (1.6). By Lemma 2.1 we have $m \ge 1$. For the proof we first construct a polynomial g with only

$m+1$ nonzero coefficients and such that $I_n(g) = I_n(f)$. This g is then used to define a polynomial h such that $I_n(h) < I_n(f)$ and we have a contradiction. The construction of h from g in [1] appears to be correct so we only consider the construction of g.

We define $g = R(f; i_0, \ldots, i_m; i_0, \ldots, i_m)$, i.e., we choose $j_0, \ldots, j_m = i_0, \ldots, i_m$ in Lemma 1.2. Then we consider for ε small enough

$$I_n(f - \varepsilon g) = \frac{\int_{-1}^{1} |f(x) - \varepsilon g(x)|\, dx}{\sum_{k=0}^{n} |d_k - \varepsilon c_k|},$$

where (c_k) are the Bernstein basis coefficients of g. In [1] we used the relation

$$|f(x) - \varepsilon g(x)| = |f(x)| - \varepsilon |g(x)| \qquad \text{for} \quad x \in [-1, 1], \tag{2.1}$$

in order to show

$$I_n(f) \le I_n(f - \varepsilon g) = \frac{\int_{-1}^{1} |f(x)|\, dx - \varepsilon \int_{-1}^{1} |g(x)|\, dx}{\sum_{k=0}^{n} |d_k| - \varepsilon \sum_{k=0}^{n} |c_k|} = \frac{I_n(f) - \varepsilon I_n(g)}{1 - \varepsilon}. \tag{2.2}$$

However, as pointed out by J. Domsta, relation (2.1) is not necessarily true for all $x \in [-1, 1]$ so that we have to use a slightly more involved argument.

Similar to the proof of Lemma 2.1 we consider for $\delta > 0$ the set

$$Z_\delta := \{x \in (-1, 1) : |f(x)| \le \delta\}.$$

Let L be an upper bound for $|g(x)|$ on $[-1, 1]$, set $\varepsilon := \delta / L$, and $f_\varepsilon := f - \varepsilon g$. Then since f and g have the same sign on $[-1, 1]$ by Lemma 1.2,

$$\begin{aligned}
\|f_\varepsilon\|_1 &= \int_{Z_\delta^c} |f(x) - \varepsilon g(x)|\, dx + \int_{Z_\delta} |f(x) - \varepsilon g(x)|\, dx \\
&\le \int_{Z_\delta^c} |f(x)|\, dx - \varepsilon \int_{Z_\delta^c} |g(x)|\, dx + \int_{Z_\delta} |f(x)|\, dx + \varepsilon \int_{Z_\delta} |g(x)|\, dx \\
&= \int_{-1}^{1} |f(x)|\, dx - \varepsilon \int_{-1}^{1} |g(x)|\, dx + 2\varepsilon \int_{Z_\delta} |g(x)|\, dx \\
&\le I_n(f) - \varepsilon I_n(g) + 2\varepsilon L \mu(Z_\delta).
\end{aligned}$$

Similarly, by (iii) in (1.7) and (1.6) we also have

$$\sum_{k=0}^{n} |d_k - \varepsilon c_k| = \sum_{k=0}^{n} |d_k| - \varepsilon \sum_{k=0}^{n} |c_k| = 1 - \varepsilon,$$

so that for $\delta = L\varepsilon$ positive and sufficiently small

$$I_n(f) \le I_n(f_\varepsilon) = \frac{\|f_\varepsilon\|_1}{\sum_{k=0}^{n} |d_k - \varepsilon c_k|} \le \frac{I_n(f) - \varepsilon I_n(g) + 2\varepsilon L \mu(Z_\delta)}{1 - \varepsilon}$$

which is an inequality similar to (2.2). Rearranging this inequality we see that $I_n(g) \leq I_n(f) + 2L\mu(Z_\delta)$. But since $\mu(Z_\delta) \to 0$ as $\delta \to 0$ this implies that $I_n(g) = I_n(f)$. ■

References

1. T. LYCHE, K. SCHERER (2002): *On the L_1-condition number of the univariate Bernstein basis*. Constr. Approx., **18**:503–528.

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