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On the p-norm condition number of the multivariate triangular Bernstein basis

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Abstract

We show that the p-norm condition number of the s-variate triangular Bernstein basis for polynomials of degree n grows at most as $O(n^s 2^n)$ for fixed s and increasing n. This is essentially the same growth as has already been established in the univariate case.

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1. Introduction

In computing with polynomials it is desirable to use a basis which is well conditioned so that small relative changes in the coefficients lead to small relative changes in the polynomial and vice versa. To measure such conditioning we use a number $\kappa_{n,s,p}$ called the p-norm condition number of the basis. For this number we use the L_p norm to measure the size of functions and the corresponding ℓ_p norm for vectors for some p with $1 \leqslant p \leqslant \infty$.

We consider here the p-norm condition number for the s-variate triangular Bernstein basis of degree n. This basis has gained increasing popularity mainly through work in computer aided geometric design [7] and it is important to know the size of its condition number.

There are good estimates for the p-norm condition number in the univariate case, in particular the exact values are known for $p = 2$ (see [3,4]) and for $p = \infty$ (see [8]). Recently, precise estimates

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for the case $p=1$ have also been given in [10], and there are improved results for univariate B-splines, see [11]. In all these cases the condition number grows like 2^n, where n is the degree of the polynomial or the piecewise polynomial.

The multivariate polynomial case for $p=\infty$ was considered in [9]. Here an upper bound for the condition number was given. For space dimension s this bound grows like $(s+1)^n$ for fixed s, but it was shown to be independent of the space dimension for $s \geqslant n$.

In this paper we determine the exact condition number in the multivariate 2-norm case and use this to give reasonably sharp estimates for any p-norm with $1 \leqslant p \leqslant \infty$. For polynomials of degree n we obtain essentially the characteristic 2^n behavior in any fixed space dimension and for any p.

The content of this paper is as follows. In Section 2 we recall the definition of the p-norm condition number and state some facts about the multivariate Bernstein basis. Most of these facts are well known and we only include short proofs. In Section 3 we study the 2-norm case. The condition number can then be computed exactly from the eigenvalues of the Gram matrix of the Bernstein basis. We show that the condition number for fixed s grows like $2^{n+s/2}/(n+s)^{1/4}$ as n increases. Some L_p inequalities in Section 4 are used to give upper and lower bounds for the condition number for $1 \leqslant p \leqslant \infty$ in Section 5. We end the paper with an appendix on the connection between Bernstein- and Legendre polynomials on simplices.

We use standard multi-index notation. Thus for tuples $\boldsymbol{j} = (j_1, \ldots, j_s)$ and $\boldsymbol{x} = (x_1, \ldots, x_s)$ we let $|\boldsymbol{j}| = j_1 + \cdots + j_s$, $\boldsymbol{j}! = j_1! j_2! \cdots j_s!$, and $\boldsymbol{x}^{\boldsymbol{j}} = x_1^{j_1} x_2^{j_2} \ldots x_s^{j_s}$.

Unless otherwise stated the indices in a sum will be nonnegative. Thus if we sum in the order $j_s, j_{s-1}, \ldots, j_1$ then

$$\sum_{|(j_1,\ldots,j_s)| \leqslant n} = \sum_{j_1=0}^{n} \sum_{j_2=0}^{n-j_1} \sum_{j_3=0}^{n-j_1-j_2} \cdots \sum_{j_s=0}^{n-j_1-\cdots-j_{s-1}} .$$

The convex hull of m points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ is denoted $\operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$, and we let $\|\boldsymbol{c}\|_p$ and $\|f\|_{L^p(\Omega)}$ be the usual p-norms of vectors and functions defined on a set Ω, respectively.

2. The Bernstein basis

For the vector space

$$P_{n,s} := P_n(\mathbb{R}^s) = \left\{ p(\boldsymbol{x}) = \sum_{|\boldsymbol{j}| \leqslant n} c_{\boldsymbol{j}} \boldsymbol{x}^{\boldsymbol{j}} \colon c_{\boldsymbol{j}} \in \mathbb{R} \right\}$$

of polynomials of total degree at most n in s variables $\boldsymbol{x} = (x_1, \ldots, x_s)$ we consider the Bernstein basis

$$\left(\frac{n!}{\boldsymbol{\alpha}!} \boldsymbol{\lambda}^{\boldsymbol{\alpha}} \right)_{|\boldsymbol{\alpha}|=n} .$$

Here $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{s+1})$ denotes the barycentric coordinate with respect to a nondegenerate simplex $\Sigma = \operatorname{conv}(\boldsymbol{v}_1, \ldots \boldsymbol{v}_{s+1})$ in $\mathbb{R}^s$ i.e., the tuple $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{x})$ corresponding to a point $\boldsymbol{x} \in \mathbb{R}^s$ is uniquely given by

$$\sum_{i=1}^{s+1} \lambda_i \boldsymbol{v}_i = \boldsymbol{x}, \qquad \sum_{i=1}^{s+1} \lambda_i = 1.$$

For $1 \leqslant p \leqslant \infty$ we define the p-norm condition number of the Bernstein basis by

$$\kappa_{n,s,p}\left(\left(\frac{n!}{\boldsymbol{\alpha}!}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\right)\right) := \sup_{c\neq 0} \frac{\| \sum_{|\alpha|=n} c_{\boldsymbol{\alpha}}(n!/\boldsymbol{\alpha}!)\boldsymbol{\lambda}^{\boldsymbol{\alpha}} \|_{L^p(\Sigma)}}{\| \boldsymbol{c} \|_p} \sup_{c\neq 0} \frac{\| \boldsymbol{c} \|_p}{\| \sum_{|\alpha|=n} c_{\boldsymbol{\alpha}}(n!/\boldsymbol{\alpha}!)\boldsymbol{\lambda}^{\boldsymbol{\alpha}} \|_{L^p(\Sigma)}}. \tag{2.1}$$

The function defined on the simplex Σ by $(x_1,\dots,x_s) \to (\lambda_1,\dots,\lambda_s)$, maps Σ onto the unit simplex

$$\Sigma_s := \mathrm{conv}(\boldsymbol{e}_1,\dots,\boldsymbol{e}_s,\boldsymbol{0}), \tag{2.2}$$

where the $\boldsymbol{e}_i, 1 \leqslant i \leqslant s$ denote the unit vectors in $\mathbb{R}^s$. Moreover, the Bernstein basis on Σ is mapped to the Bernstein basis on Σ_s. Denoting these functions by $(B^n_{\boldsymbol{j}})$ it follows that

$$\kappa_{n,s,p} := \kappa_{n,s,p}((n!\boldsymbol{\lambda}^{\boldsymbol{\alpha}}/\boldsymbol{\alpha}!)) = \kappa_{n,s,p}((B^n_{\boldsymbol{j}}))$$

and it is enough to study the condition number problem on the unit simplex Σ_s.

In the rest of this section we recall some elementary facts about Bernstein basis functions on the unit simplex. The functions $B^n_{\boldsymbol{j}}$ can be written in the form

$$B^n_{j_1,\dots,j_s}(x_1,\dots,x_s) = \frac{n!}{j_1!\cdots j_s!(n-j_1-\cdots-j_s)!} x_1^{j_1}\cdots x_s^{j_s}(1-x_1-\cdots-x_s)^{n-j_1-\cdots-j_s},$$

or more compactly using multi-index notation

$$B^n_{\boldsymbol{j}}(\boldsymbol{x}) = \begin{bmatrix} n \\ \boldsymbol{j} \end{bmatrix} \boldsymbol{x}^{\boldsymbol{j}}(1-|\boldsymbol{x}|)^{n-|\boldsymbol{j}|} \quad \boldsymbol{j} \geqslant 0 \quad |\boldsymbol{j}| \leqslant n, \tag{2.3}$$

where $\boldsymbol{j} = (j_1,\dots,j_s)$, $\boldsymbol{x} = (x_1,\dots,x_s)$, and

$$\begin{bmatrix} n \\ \boldsymbol{j} \end{bmatrix} = \frac{n!}{\boldsymbol{j}!(n-|\boldsymbol{j}|)!}$$

is a multinomial coefficient.

The following well-known elementary properties of the Bernstein basis will be useful.

Lemma 1. *We have*

(1) $B^n_{\boldsymbol{j}}(\boldsymbol{x}) > 0$ *for* $|\boldsymbol{j}| \leqslant n$ *and all* $\boldsymbol{x} \in \Sigma_s$.

(2) $\sum_{|\boldsymbol{j}|\leqslant n} B^n_{\boldsymbol{j}}(\boldsymbol{x}) = 1$ *for all* $\boldsymbol{x} \in \mathbb{R}^s$.

(3) $\int_{\Sigma_s} \boldsymbol{x}^{\boldsymbol{r}} B^n_{\boldsymbol{j}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \dfrac{(\boldsymbol{j}+\boldsymbol{r})!n!}{\boldsymbol{j}!(n+s+|\boldsymbol{r}|)!}, \quad |\boldsymbol{r}| \leqslant n.$

(4) $\dfrac{n!}{(n-|\boldsymbol{r}|)!}\boldsymbol{x}^{\boldsymbol{r}} = \sum_{|\boldsymbol{j}|\leqslant n} \dfrac{\boldsymbol{j}!}{(\boldsymbol{j}-\boldsymbol{r})!} B^n_{\boldsymbol{j}}(\boldsymbol{x}), \qquad |\boldsymbol{r}| \leqslant n.$

Proof. Statement (1) follows immediately from Eq. (2.3) since

$$\Sigma_s = \left\{(x_1,\dots,x_s) : x_j \geqslant 0 \text{ all } j \text{ and } \sum_j x_j \leqslant 1\right\},$$

while (2) is a special case of (4). For (3) it is well known (see [2, p. 140]) that

$$\int_{\Sigma_s} B_j^n(x)\,\mathrm{d}x = \frac{n!}{(n+s)!}, \quad |\boldsymbol{j}| \leqslant n.$$

Combining this with

$$\int_{\Sigma_s} \boldsymbol{x}^{\boldsymbol{r}} B_{\boldsymbol{j}}^n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \frac{(\boldsymbol{r}+\boldsymbol{j})!}{\boldsymbol{j}!}\frac{n!}{(n+|\boldsymbol{r}|)!}\int_{\Sigma_s} B_{\boldsymbol{r}+\boldsymbol{j}}^{n+|\boldsymbol{r}|}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

we obtain (3). To prove (4) we use generating functions. Differentiating the relation

$$(1-|\boldsymbol{x}|+\boldsymbol{t}\cdot\boldsymbol{x})^n := \sum_{i=0}^{n}\binom{n}{i}(1-|\boldsymbol{x}|)^{n-i}(\boldsymbol{t}\cdot\boldsymbol{x})^i = \sum_{|\boldsymbol{j}|\leqslant n}\boldsymbol{t}^{\boldsymbol{j}} B_{\boldsymbol{j}}^n(\boldsymbol{x}), \quad \boldsymbol{t},\boldsymbol{x}\in\mathbb{R}^s$$

$\boldsymbol{r}$ times with respect to $\boldsymbol{t}$ and then setting $\boldsymbol{t}=(1,\dots,1)$ we obtain (4). □

3. The L_2 case

In this section we give an exact formula and asymptotic estimates for the L_2 condition number

$$\kappa_{n,s,2} = \sup_{(c_j)\neq 0}\frac{\|\sum_{|\boldsymbol{j}|\leqslant n} c_{\boldsymbol{j}} B_{\boldsymbol{j}}^n\|_{L_2(\Sigma_s)}}{\|(c_{\boldsymbol{j}})\|_2}\sup_{(c_j)\neq 0}\frac{\|(c_{\boldsymbol{j}})\|_2}{\|\sum_{|\boldsymbol{j}|\leqslant n} c_{\boldsymbol{j}} B_{\boldsymbol{j}}^n\|_{L_2(\Sigma_s)}}. \tag{3.1}$$

To start, we observe that

$$\kappa_{n,s,2} = \sup_{\boldsymbol{c}\neq 0}\frac{\sqrt{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{c}}}{\|\boldsymbol{c}\|_2}\Bigg/\inf_{\boldsymbol{c}\neq 0}\frac{\sqrt{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{c}}}{\|\boldsymbol{c}\|_2} = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}},$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalue of the Gram matrix $\boldsymbol{G}$ of the Bernstein basis

$$\boldsymbol{G} = (\langle B_{\boldsymbol{i}}^n, B_{\boldsymbol{j}}^n\rangle)_{|\boldsymbol{i}|,|\boldsymbol{j}|\leqslant n} = \left(\int_{\Sigma_s} B_{\boldsymbol{i}}^n(\boldsymbol{x}) B_{\boldsymbol{j}}^n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)_{|\boldsymbol{i}|,|\boldsymbol{j}|\leqslant n}. \tag{3.2}$$

This is a matrix of order $\binom{n+s}{s}$ and for both rows and columns we use the linear ordering of s-tuples $\boldsymbol{i},\boldsymbol{j}$ given by $\boldsymbol{i}>\boldsymbol{j}$ if and only if the first nonzero component of $\boldsymbol{i}-\boldsymbol{j}$ is positive. As an example, for $n=s=2$ the lower indexes of the basis functions will be taken in the order $(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)$.

The eigenvalues of the Gram matrix can be determined explicitly.

Theorem 2. *The Gram matrix* (3.2) *of the s-variate triangular Bernstein basis of degree n has the eigenvalues*

$$\lambda_{\boldsymbol{m}} = \frac{(n-|\boldsymbol{m}|+1)\cdots(n-1)n}{(n+1)(n+2)\cdots(n+|\boldsymbol{m}|+s)}, \quad |\boldsymbol{m}|\leqslant n. \tag{3.3}$$

Proof. We consider polynomials $Q_{\boldsymbol{m}}$ (to be determined) of the special form

$$Q_{\boldsymbol{m}}(x) = \sum_{|\boldsymbol{r}|\leqslant|\boldsymbol{m}|} q_{\boldsymbol{m},\boldsymbol{r}}\boldsymbol{x}^{\boldsymbol{r}}, \quad \boldsymbol{m}\in\mathbb{Z}^s \quad \text{with} \quad \boldsymbol{m}\geqslant 0, \quad \text{and} \quad |\boldsymbol{m}|\leqslant n, \tag{3.4}$$

where

$$q_{\boldsymbol{m},\boldsymbol{m}} = 1, \quad \text{and} \quad q_{\boldsymbol{m},\boldsymbol{r}} = 0 \quad \text{for} \quad |\boldsymbol{r}| = |\boldsymbol{m}| \quad \text{and} \quad \boldsymbol{r} \neq \boldsymbol{m}. \tag{3.5}$$

Note that $Q_{\boldsymbol{m}}$ is a polynomial of degree $|\boldsymbol{m}|$. Let $\boldsymbol{d}_{\boldsymbol{m}} = (d_{\boldsymbol{m},\boldsymbol{i}})$ be the degree n vector of BB-coefficients of $Q_{\boldsymbol{m}}$ so that

$$Q_{\boldsymbol{m}}(x) = \sum_{|\boldsymbol{i}| \leqslant n} d_{\boldsymbol{m},\boldsymbol{i}} B_{\boldsymbol{i}}^n(\boldsymbol{x}), \quad |\boldsymbol{m}| \leqslant n.$$

We will determine for fixed $\boldsymbol{m}$ the coefficients $q_{\boldsymbol{m},\boldsymbol{r}}$ so that $\boldsymbol{d}_{\boldsymbol{m}}$ is an eigenvector of $\boldsymbol{G}$. The idea is to express both $\boldsymbol{d}_{\boldsymbol{m}}$ and $\boldsymbol{G}\boldsymbol{d}_{\boldsymbol{m}}$ in terms of the $q_{\boldsymbol{m},\boldsymbol{r}}$ and use the eigenvalue/eigenvector relation $\boldsymbol{G}\boldsymbol{d}_{\boldsymbol{m}} = \lambda_{\boldsymbol{m}}\boldsymbol{d}_{\boldsymbol{m}}$ to determine both $\lambda_{\boldsymbol{m}}$ and $Q_{\boldsymbol{m}}$. Consider first the vector $\boldsymbol{d}_{\boldsymbol{m}}$. Inserting (4) of Lemma 1 into (3.4) we can express each $d_{\boldsymbol{m},\boldsymbol{i}}$ in the form

$$d_{\boldsymbol{m},\boldsymbol{i}} = \sum_{|\boldsymbol{r}| \leqslant |\boldsymbol{m}|} q_{\boldsymbol{m},\boldsymbol{r}} \frac{(n - |\boldsymbol{r}|)!\boldsymbol{i}!}{n!(\boldsymbol{i} - \boldsymbol{r})!} = \sum_{|\boldsymbol{r}| \leqslant |\boldsymbol{m}|} q_{\boldsymbol{m},\boldsymbol{r}} \sum_{\boldsymbol{j} \leqslant \boldsymbol{r}} \beta_{\boldsymbol{r},\boldsymbol{j}} \boldsymbol{i}^{\boldsymbol{j}} \tag{3.6}$$

for some constants $\beta_{\boldsymbol{r},\boldsymbol{i}}$ independent of $\boldsymbol{i}$, in particular $\beta_{\boldsymbol{r},\boldsymbol{r}} = (n - |\boldsymbol{r}|)!/n!$. Similarly, using (3) of Lemma 1 we can express the $\boldsymbol{i}$th component of $\boldsymbol{G}\boldsymbol{d}_{\boldsymbol{m}}$ in the form

$$\begin{aligned}(\boldsymbol{G}\boldsymbol{d}_{\boldsymbol{m}})_{\boldsymbol{i}} &= \langle B_{\boldsymbol{i}}^n, Q_{\boldsymbol{m}} \rangle = \sum_{|\boldsymbol{r}| \leqslant |\boldsymbol{m}|} q_{\boldsymbol{m},\boldsymbol{r}} \langle B_{\boldsymbol{i}}^n, \boldsymbol{x}^{\boldsymbol{r}} \rangle \\ &= \sum_{|\boldsymbol{r}| \leqslant |\boldsymbol{m}|} q_{\boldsymbol{m},\boldsymbol{r}} \frac{(\boldsymbol{i} + \boldsymbol{r})!n!}{\boldsymbol{i}!(n + s + |\boldsymbol{r}|)!} = \sum_{|\boldsymbol{r}| \leqslant |\boldsymbol{m}|} q_{\boldsymbol{m},\boldsymbol{r}} \sum_{\boldsymbol{j} \leqslant \boldsymbol{r}} \alpha_{\boldsymbol{r},\boldsymbol{j}} \boldsymbol{i}^{\boldsymbol{j}},\end{aligned} \tag{3.7}$$

where the $\alpha_{\boldsymbol{r},\boldsymbol{j}}$ are independent of $\boldsymbol{i}$, in particular $\alpha_{\boldsymbol{r},\boldsymbol{r}} = n!/(n + s + |\boldsymbol{r}|)!$. We will need the value of the following ratio:

$$\frac{\alpha_{\boldsymbol{r},\boldsymbol{r}}}{\beta_{\boldsymbol{r},\boldsymbol{r}}} = \frac{(n!)^2}{(n + s + |\boldsymbol{r}|)!(n - |\boldsymbol{r}|)!}, \quad |\boldsymbol{r}| \leqslant |\boldsymbol{m}|. \tag{3.8}$$

Switching the order of summation in (3.6) and (3.7) we have $\boldsymbol{G}\boldsymbol{d}_{\boldsymbol{m}} = \lambda_{\boldsymbol{m}}\boldsymbol{d}_{\boldsymbol{m}}$ if and only if

$$\sum_{|\boldsymbol{j}| \leqslant |\boldsymbol{m}|} \sum_{\substack{\boldsymbol{r} \geqslant \boldsymbol{j} \\ |\boldsymbol{r}| \leqslant |\boldsymbol{m}|}} q_{\boldsymbol{m},\boldsymbol{r}} [\alpha_{\boldsymbol{r},\boldsymbol{j}} - \lambda_{\boldsymbol{m}} \beta_{\boldsymbol{r},\boldsymbol{j}}] \boldsymbol{i}^{\boldsymbol{j}} = 0, \quad |\boldsymbol{i}| \leqslant n.$$

We see that this holds for all such $\boldsymbol{i}$ if and only if

$$\sum_{\substack{\boldsymbol{r} \geqslant \boldsymbol{j} \\ |\boldsymbol{r}| \leqslant |\boldsymbol{m}|}} q_{\boldsymbol{m},\boldsymbol{r}} [\alpha_{\boldsymbol{r},\boldsymbol{j}} - \lambda_{\boldsymbol{m}} \beta_{\boldsymbol{r},\boldsymbol{j}}] = 0, \quad |\boldsymbol{j}| \leqslant |\boldsymbol{m}|. \tag{3.9}$$

Since $q_{\boldsymbol{m},\boldsymbol{m}} = 1$ we obtain the value $\lambda_{\boldsymbol{m}} = (\alpha_{\boldsymbol{m},\boldsymbol{m}}/\beta_{\boldsymbol{m},\boldsymbol{m}})$ by choosing $\boldsymbol{j} = \boldsymbol{m}$ in this equation, and (3.3) follows by letting $\boldsymbol{r} = \boldsymbol{m}$ in (3.8). The coefficients $q_{\boldsymbol{m},\boldsymbol{r}}$ can now be determined in a triangular fashion from (3.9). We already know $q_{\boldsymbol{m},\boldsymbol{r}}$ for $|\boldsymbol{r}| = |\boldsymbol{m}|$ from (3.5) and if $q_{\boldsymbol{m},\boldsymbol{r}}$ has been determined for $\boldsymbol{r} \geqslant \boldsymbol{j}$ with $\boldsymbol{r} \neq \boldsymbol{j}$ then we determine $q_{\boldsymbol{m},\boldsymbol{j}}$ from (3.9). It suffices to make sure that the coefficient $\alpha_{\boldsymbol{j},\boldsymbol{j}} - \lambda_{\boldsymbol{m}}\beta_{\boldsymbol{j},\boldsymbol{j}}$ in front of $q_{\boldsymbol{m},\boldsymbol{j}}$ does not vanish. This follows since from (3.8)

$$\frac{\alpha_{\boldsymbol{j},\boldsymbol{j}}}{\beta_{\boldsymbol{j},\boldsymbol{j}}} \neq \frac{\alpha_{\boldsymbol{m},\boldsymbol{m}}}{\beta_{\boldsymbol{m},\boldsymbol{m}}} = \lambda_{\boldsymbol{m}} \quad \text{for} \quad |\boldsymbol{j}| < |\boldsymbol{m}|. \qquad \square$$

In the appendix we will point out that the eigenpolynomials $Q_{\boldsymbol{m}}(x)$ defined by (3.4) can be identified as Legendre polynomials with respect to the simplex Σ_s (cf. [1]). This result was already shown by Deriennic [5], who obtains the polynomials $Q_{\boldsymbol{m}}(x)$ as eigenfunctions of the L_2-projection operator with respect to the Bernstein–Bézier basis, see also [3]. However our approach seems to be more direct in view of our actual goal of determining the L_2 condition number and therefore we have present it here.

Using the explicit formulae for the eigenvalues of the Gram matrix we find the exact formula for the 2-norm condition number. The following theorem generalizes the univariate result proved in [4] to the multivariate case.

Theorem 3. *For any $n, s \geqslant 1$ the L_2 condition number $\kappa_{n,s,2}$ of the triangular s-dimensional Bernstein–Bézier basis of degree n is given by*

$$\kappa_{n,s,2} = \sqrt{\binom{2n+s}{n}}. \tag{3.10}$$

Moreover, we have the lower and upper bounds

$$\exp\left\{\frac{-s(s-1)}{8n}\right\} \frac{2^{n+s/2}}{(\pi(n+s+1/2))^{1/4}} \leqslant \kappa_{n,s,2} \leqslant \frac{2^{n+s/2}}{(\pi(n+s))^{1/4}} \exp\left\{\frac{-s(s-1)}{8(n+s)}\right\}. \tag{3.11}$$

Proof. The largest and smallest eigenvalue of the Gram matrix are given by

$$\lambda_{\max} = \frac{n!}{(n+s)!}, \quad \lambda_{\min} = \frac{(n!)^2}{(2n+s)!}$$

which represent the cases $|\boldsymbol{m}| = 0$ and $|\boldsymbol{m}| = n$ in (3.3). But then

$$\kappa_{n,s,2} = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} = \sqrt{\frac{(2n+s)!}{n!(n+s)!}} = \sqrt{\binom{2n+s}{n}},$$

which proves (3.10).

Consider next the asymptotic estimate. We have

$$\binom{2n+s}{n} = 2^{-s}\binom{2n+2s}{n+s}\prod_{v=1}^{s-1}\frac{2n+2v}{2n+s+v} \tag{3.12}$$

and the binomial coefficient is bounded below and above by Wallis' inequality

$$\frac{2^{2n+2s}}{\sqrt{\pi(n+s+1/2)}} \leqslant \binom{2n+2s}{n+s} \leqslant \frac{2^{2n+2s}}{\sqrt{\pi(n+s)}}. \tag{3.13}$$

For the product term we use the inequality $1+x\leqslant e^x$, valid for all x to obtain

$$\prod_{v=1}^{s-1}\frac{2n+s+v}{2n+2v}\leqslant\prod_{v=1}^{s-1}\left(1+\frac{s-v}{2n}\right)\leqslant\prod_{v=1}^{s-1}\exp\left\{\frac{s-v}{2n}\right\}=\exp\left\{\frac{s(s-1)}{4n}\right\}$$

and with $x_v=(s-v)/(2(n+s))$

$$\prod_{v=1}^{s-1}\frac{2n+2v}{2n+s+v}=\prod_{v=1}^{s-1}\frac{1-2x_v}{1-x_v}\leqslant\prod_{v=1}^{s-1}(1-x_v)\leqslant\prod_{v=1}^{s-1}\exp\{-x_v\}=\exp\left\{-\frac{s(s-1)}{4(n+s)}\right\}.$$

Combining these inequalities we find

$$\exp\left\{\frac{-s(s-1)}{4n}\right\}\leqslant\prod_{v=1}^{s-1}\frac{2n+2v}{2n+s+v}\leqslant\exp\left\{\frac{-s(s-1)}{4(n+s)}\right\},\tag{3.14}$$

and inserting (3.13) and (3.14) in (3.12) result in the following bounds:

$$\exp\left\{-\frac{s(s-1)}{4n}\right\}\frac{2^{2n+s}}{\sqrt{\pi(n+s+1/2)}}\leqslant\binom{2n+s}{n}\leqslant\frac{2^{2n+s}}{\sqrt{\pi(n+s)}}\exp\left\{-\frac{s(s-1)}{4(n+s)}\right\}.$$

Inequalities (3.11) now follow by taking square roots. □

In [9] it was shown that $\kappa_{n,s,\infty}$ could be bounded independently of s for $s\geqslant n$. Formula (3.10) shows that such a bound does not hold for $p=2$.

It is interesting to determine the extremal coefficients in (3.1) more explicitly. The first sup is, appart from scaling, uniquely attained for the eigenpolynomial $Q_0(x)=1$ corresponding to the eigenvalue $\lambda_{\max}$. The corresponding extremal coefficients are given by $c_i=1$ for all i. The second sup is more complicated. For any $\boldsymbol{m}\in\mathbb{Z}^s$ with $\boldsymbol{m}\geqslant 0$ and $|\boldsymbol{m}|=n$ the vectors $\boldsymbol{c_m}=(c_{\boldsymbol{m},\boldsymbol{j}})_{|\boldsymbol{j}|\leqslant n}$ given by

$$c_{\boldsymbol{m},\boldsymbol{j}}=\begin{cases}(-1)^{|\boldsymbol{j}|}\binom{\boldsymbol{m}}{\boldsymbol{j}} & \text{if } \boldsymbol{j}\leqslant\boldsymbol{m},\\ 0 & \text{otherwise}\end{cases}\tag{3.15}$$

are a collection of $\binom{n+s-1}{s-1}$ linearly independent extremal vectors for the second sup in (3.1). The corresponding extremal polynomials are the classical [1] Legendre polynomials on Σ_s of degree $|\boldsymbol{m}|=n$, see the appendix for details.

4. Some p-norm inequalities

In this section we give L_p inequalities which will be used to relate the condition numbers for different p.

We start with some inequalities bounding the size of vectors and functions in different p-norms.

Lemma 4. *For a vector* $\mathbf{c} \in \mathbb{R}^m$ *we have the following inequality*:

$$\| \mathbf{c} \|_p \leqslant \| \mathbf{c} \|_q \leqslant m^{1/q-1/p} \| \mathbf{c} \|_p, \quad 1 \leqslant q \leqslant p \leqslant \infty. \tag{4.1}$$

Suppose for some bounded subset $\Omega \subset \mathbb{R}^s$ *and a function* $f \in L_1(\Omega)$ *we can bound the* L_∞ *norm in terms of the* L_1 *norm*

$$\| f \|_{L_\infty(\Omega)} \leqslant \gamma \| f \|_{L_1(\Omega)} \tag{4.2}$$

for some $\gamma > 0$. *Then the following inequalities hold*:

$$\frac{1}{\gamma^{1/q-1/p}} \| f \|_{L_p(\Omega)} \leqslant \| f \|_{L_q(\Omega)} \leqslant vol(\Omega)^{1/q-1/p} \| f \|_{L_p(\Omega)}, \quad 1 \leqslant q \leqslant p \leqslant \infty. \tag{4.3}$$

Proof. The leftmost inequality of (4.1) follows from Jensens inequality, while the rightmost one is a standard application of Holders inequality. In the proof of (4.3) we use $\| f \|_p$ as an abbreviation for $\| f \|_{L_p(\Omega)}$ for any $1 \leqslant p \leqslant \infty$. Let

$$T_1 : L_1(\Omega) \to L_\infty(\Omega), \quad T_2 : L_\infty(\Omega) \to L_\infty(\Omega), \quad T_3 : L_q(\Omega) \to L_\infty(\Omega)$$

all be defined as the identity operator between the indicated spaces. By the Riesz–Thorin interpolation Theorem, see [6, p. 32], and (4.2) we obtain

$$\frac{\| f \|_\infty}{\| f \|_q} \leqslant \| T_3 \| \leqslant \| T_1 \|^{1/q} \| T_2 \|^{1-1/q} \leqslant \gamma^{1/q}.$$

Hence the leftmost inequality in (4.3) follows for $p = \infty$ and any q. We extend this inequality to any $p \geqslant q$ by the string of inequalities

$$\| f \|_p^p = \int |f|^{p-q} |f|^q \leqslant \| f \|_\infty^{p-q} \| f \|_q^q \leqslant (\gamma^{1/q} \| f \|_q)^{p-q} \| f \|_q^q = \gamma^{p/q-1} \| f \|_q^p .$$

Taking pth roots completes the proof of the leftmost inequality. For the rightmost inequality we have by the Holder

$$\| f \|_q^q = \int |f|^q \leqslant \left(\int (|f|^q)^{p/q} \right)^{q/p} \left(\int 1 \right)^{1-q/p} .$$

We now take qth roots. □

To estimate the constant γ in (4.2) in the polynomial case on Σ_s we need a version of Markov's inequality.

Lemma 5. *For positive integers* n, s *and* $f \in P_n(\mathbb{R}^s)$ *we have*

$$\| \nabla f \|_{L_\infty(\Sigma_s)} \leqslant 4n^2 \sqrt{s} \| f \|_{L_\infty(\Sigma_s)}, \tag{4.4}$$

where

$$\| \nabla f \|_{L_\infty(\Sigma_s)} = \max_{1 \leqslant i \leqslant s} \left\| \frac{\partial f}{\partial x_i} \right\|_{L_\infty(\Sigma_s)}.$$

Proof. By [13]

$$\left\| \left(\sum_{i=1}^{s} \left(\frac{\partial f}{\partial x_i} \right)^2 \right)^{1/2} \right\|_{L_\infty(\Sigma_s)} \leqslant \frac{4n^2}{w} \| f \|_{L_\infty(\Sigma_s)},$$

where w is the minimal distance between two parallel supporting hyperplanes containing Σ_s between them. Since for each $\boldsymbol{x} \in \Sigma_s$ we have

$$\| \nabla f(\boldsymbol{x}) \|_\infty \leqslant \| \nabla f(\boldsymbol{x}) \|_2 = \left(\sum_{i=1}^{s} \left(\frac{\partial f(\boldsymbol{x})}{\partial x_i} \right)^2 \right)^{1/2},$$

we obtain the result if we can show that $w = 1/\sqrt{s}$. To compute w it is sufficient to consider $s+1$ hyperplanes $H_1, \ldots, H_s$ where each H_i contains the facet

$$\Sigma_s^i = \mathrm{conv}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{i-1}, \boldsymbol{e}_{i+1}, \ldots, \boldsymbol{e}_{s+1}),$$

where $\boldsymbol{e}_{s+1} = \boldsymbol{0}$. The parallel supporting hyperplane K_i containing Σ_s between H_i and K_i must pass through $\boldsymbol{e}_i$ and the distance between H_i and K_i is given by

$$\inf_{\boldsymbol{x} \in \Sigma_s^i} \left\| \boldsymbol{e}_i - \sum_{\substack{j=1 \\ j \neq i}}^{s+1} x_j \boldsymbol{e}_j \right\|_2.$$

Now

$$\boldsymbol{x} \in \Sigma_s^i \Leftrightarrow \sum_{\substack{j=1 \\ j \neq i}}^{s+1} x_j = 1, \quad x_j \geqslant 0$$

and the inf for $1 \leqslant i \leqslant s$ is equal to one and is obtained for $\boldsymbol{x}$ located at the origin. If $i = s+1$ then the inf is equal to $1/\sqrt{s}$ and is obtained for $x = (1/s, \ldots, 1/s)$. □

We can now give an estimate for the constant γ in Lemma 4 when Ω is the unit simplex in $\mathbb{R}^s$ and f is a polynomial of degree n. For a similar result in the univariate case see [12].

Lemma 6. *For $n \geqslant s^{1/4}/2$ and any $f \in P_n(\mathbb{R}^s)$ we have*

$$\| f \|_{L_\infty(\Sigma_s)} \leqslant s! K_s n^{2s} \| f \|_{L_1(\Sigma_s)}, \tag{4.5}$$

where

$$K_s = \mathrm{e}^{-1/2}(s+1) 8^s s^{s/2}. \tag{4.6}$$

Proof. Let $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_s)\in\Sigma_s$ be a point where f attains its norm, i.e.,

$$\| f \|_{L_\infty(\Sigma_s)} = |f(\boldsymbol{\alpha})|.$$

We define α_{s+1} so that $\sum_{j=1}^{s+1}\alpha_j=1$, and as usual we set $\boldsymbol{e}_{s+1}=\boldsymbol{0}$, the zero vector. Clearly, $\alpha_i\geqslant 1/(s+1)$ for some i with $1\leqslant i\leqslant s+1$, and for this i and any $\mu>0$ we consider the simplex

$$\Sigma_{s,\mu}^{(i)}(\boldsymbol{\alpha}) = \{\boldsymbol{\alpha}+\mu(\boldsymbol{\lambda}-\boldsymbol{e}_i) : \boldsymbol{\lambda}\in\Sigma_s\}. \tag{4.7}$$

We note that $\Sigma_{s,\mu}^{(i)}(\boldsymbol{\alpha})\subset\Sigma_s$ provided $0<\mu\leqslant\alpha_i$.

Fix $\boldsymbol{x}\in\Sigma_{s,\mu}^{(i)}(\boldsymbol{\alpha})$. By the chain rule and the Markov inequality we have

$$\begin{aligned}|f(\boldsymbol{\alpha})|-|f(\boldsymbol{x})| &\leqslant |f(\boldsymbol{\alpha})-f(\boldsymbol{x})|\\ &\leqslant \| \nabla f \|_{L_\infty(\Sigma_s)} \| \boldsymbol{\alpha}-\boldsymbol{x} \|_1\\ &\leqslant M|f(\boldsymbol{\alpha})| \, \| \boldsymbol{\alpha}-\boldsymbol{x} \|_1,\end{aligned}$$

where $M=4n^2\sqrt{s}$ is the constant in the Markov inequality (4.4). Rearranging this inequality and integrating we find

$$\| f \|_{L_\infty(\Sigma_s)} \int_{\Sigma_{s,\mu}^{(i)}(\boldsymbol{\alpha})} (1-M\| \boldsymbol{\alpha}-\boldsymbol{x} \|_1)\,\mathrm{d}\boldsymbol{x} \leqslant \int_{\Sigma_{s,\mu}^{(i)}(\boldsymbol{\alpha})} |f(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x}.$$

Since $\alpha_i\geqslant 1/(s+1)$ and $\Sigma_{s,\mu}^{(i)}(\boldsymbol{\alpha})\subset\Sigma_s$ provided $0<\mu\leqslant\alpha_i$ it follows that

$$\| f \|_{L_\infty(\Sigma_s)} \leqslant \min_{0<\mu\leqslant 1/(s+1)} \max_{1\leqslant i\leqslant s+1} \frac{1}{g_i(\mu)} \| f \|_{L_1(\Sigma_s)}, \tag{4.8}$$

where

$$g_i(\mu)=\int_{\Sigma_{s,\mu}^{(i)}(\boldsymbol{\alpha})}(1-M\| \boldsymbol{\alpha}-\boldsymbol{x} \|_1)\,\mathrm{d}\boldsymbol{x} = \mu^s\int_{\Sigma_s}(1-\mu M\| \boldsymbol{\lambda}-\boldsymbol{e}_i \|_1)\,\mathrm{d}\boldsymbol{\lambda}. \tag{4.9}$$

To evaluate these integrals we observe that

$$\int_{\Sigma_s} 1\,\mathrm{d}\boldsymbol{\lambda}=\frac{1}{s!}, \qquad \int_{\Sigma_s}\lambda_j\,\mathrm{d}\boldsymbol{\lambda}=\frac{1}{(s+1)!}, \quad j=1,\dots,s,$$

so that for $1\leqslant i\leqslant s$

$$\begin{aligned}g_i(\mu) &= \mu^s\int_{\Sigma_s}\left(1-\mu M\left(1-2\lambda_i+\sum_{j=1}^{s}\lambda_j\right)\right)\mathrm{d}\boldsymbol{\lambda}\\ &= \mu^s\left(\frac{1}{s!}-\mu M\left(\frac{1}{s!}-\frac{2}{(s+1)!}+\frac{s}{(s+1)!}\right)\right)\\ &= \frac{\mu^s}{s!}\left(1-\mu M\frac{2s-1}{s+1}\right).\end{aligned}$$

For $\mu>0$ the function g_i has a unique maximum at $\mu=\mu^*$ given by

$$\mu^*=\frac{s}{M(2s-1)}$$

with corresponding value

$$\frac{1}{g_i(\mu^*)}=\left(\frac{2s-1}{s}\right)^s(s+1)!M^s, \quad i=1,\dots,s. \tag{4.10}$$

For $i = s+1$ we find

$$g_{s+1}(\mu) = \mu^s \int_{\Sigma_s} \left(1 - \mu M \sum_{j=1}^{s} \lambda_j\right) \mathrm{d}\boldsymbol{\lambda}$$
$$= \frac{\mu^s}{s!}\left(1 - \mu M \frac{s}{s+1}\right)$$

and we see that $g_{s+1}(\mu^*) > g_i(\mu^*)$ for $i=1,\ldots,s$. The condition $n > s^{1/4}/2$ implies that $\mu^* \leqslant 1/(s+1)$ and we have shown that for $n \geqslant s^{1/4}/2$ the solution of the min–max problem (4.8) is given by the value in (4.10). Since $((2s-1)/s)^s \leqslant \mathrm{e}^{-1/2}2^s$ and $M = 4n^2\sqrt{s}$ we obtain the estimate in (4.5). □

5. Estimates for general L_p-norms

Consider the condition number of the Bernstein basis on the unit simplex Σ_s

$$\kappa_{n,s,p} = \sup_{(c_j)\neq 0} \frac{\| \sum_{|\boldsymbol{j}|\leqslant n} c_{\boldsymbol{j}} B^n_{\boldsymbol{j}} \|_{L^p(\Sigma_s)}}{\| (c_{\boldsymbol{j}}) \|_p} \sup_{(c_j)\neq 0} \frac{\| (c_{\boldsymbol{j}}) \|_p}{\| \sum_{|\boldsymbol{j}|\leqslant n} c_{\boldsymbol{j}} B^n_{\boldsymbol{j}} \|_{L^p(\Sigma_s)}}. \tag{5.1}$$

The first factor can be computed exactly for any p.

Lemma 7. *For* $1 \leqslant p \leqslant \infty$

$$\sup_{(c_j)\neq 0} \frac{\| \sum_{|\boldsymbol{j}|\leqslant n} c_{\boldsymbol{j}} B^n_{\boldsymbol{j}} \|_{L^p(\Sigma_s)}}{\| (c_{\boldsymbol{j}}) \|_p} = \left(\frac{v}{m}\right)^{1/p}, \tag{5.2}$$

where

$$m = \dim(P_n(\mathbb{R}^s)) = \binom{n+s}{s}, \quad \text{and} \quad v = \mathrm{vol}(\Sigma_s) = \frac{1}{s!}.$$

Proof. Using Lemma 1 and the Holder inequality with $1/p + 1/q = 1$ we obtain

$$\left\| \sum_{|\boldsymbol{j}|\leqslant n} c_{\boldsymbol{j}} B^n_{\boldsymbol{j}} \right\|^p_{L^p(\Sigma_s)} = \int_{\Sigma_s} \left| \sum_{\boldsymbol{j}} c_{\boldsymbol{j}} B^n_{\boldsymbol{j}}(\boldsymbol{x})^{1/p} B^n_{\boldsymbol{j}}(\boldsymbol{x})^{1/q} \right|^p \mathrm{d}\boldsymbol{x}$$
$$\leqslant \int_{\Sigma_s} \left(\sum_{\boldsymbol{j}} |c_{\boldsymbol{j}}|^p B^n_{\boldsymbol{j}}(\boldsymbol{x}) \right) \left(\sum_{\boldsymbol{j}} B^n_{\boldsymbol{j}}(\boldsymbol{x}) \right)^{p/q} \mathrm{d}\boldsymbol{x}$$
$$= \frac{v}{m} \| (c_{\boldsymbol{j}}) \|^p_p .$$

Taking pth roots we obtain (5.2) with an inequality. However, we obtain equality by the choice $c^*_{\boldsymbol{j}} = 1$ for all $\boldsymbol{j}$. Indeed, since $\sum_{|\boldsymbol{j}|\leqslant n} c^*_{\boldsymbol{j}} B^n_{\boldsymbol{j}} = 1$ we then have

$$\frac{\| \sum_{|\boldsymbol{j}|\leqslant n} c^*_{\boldsymbol{j}} B^n_{\boldsymbol{j}} \|_{L^p(\Sigma_s)}}{\| (c^*_{\boldsymbol{j}}) \|_p} = \frac{v^{1/p}}{m^{1/p}}. \qquad □$$

Theorem 8. *For $n,s\geqslant 1$ and $1\leqslant q\leqslant p\leqslant\infty$*

$$\frac{1}{(K_s n^{2s})^{1/q-1/p}}\kappa_{n,s,q}\leqslant\kappa_{n,s,p}\leqslant\binom{n+s}{s}^{1/q-1/p}\kappa_{n,s,q}, \tag{5.3}$$

where K_s given by (4.6) *only depends on s.*

Proof. By Lemma 7 we have for any coefficients $\boldsymbol{c}=(c_j)\neq 0$ and $f=\sum c_j B_j^n$

$$\frac{\| f\|_p}{\|\boldsymbol{c}\|_p}=\left(\frac{m}{v}\right)^{1/q-1/p}\frac{\| f\|_q}{\|\boldsymbol{c}\|_q}. \tag{5.4}$$

From the bounds in Lemma 4 it follows that

$$\frac{1}{(m\gamma)^{1/q-1/p}}\frac{\|\boldsymbol{c}\|_q}{\| f\|_q}\leqslant\frac{\|\boldsymbol{c}\|_p}{\| f\|_p}\leqslant\left(\frac{v}{m}\right)^{1/q-1/p}\frac{\|\boldsymbol{c}\|_q}{\| f\|_q},$$

where $\gamma=s!K_s n^{2s}=K_s n^{2s}/v$ is the constant in (4.6). Taking the supremum in this inequality and in (5.4) it is an easy matter to complete the proof. □

This estimate shows that the condition numbers $\kappa_{n,s,p}(\Sigma_s)$ differ with respect to p only by a rational factor in n. More precisely, we have

Corollary 9. *For $n,s\geqslant 1$ and $1\leqslant p\leqslant\infty$ we have the estimates*

$$K_1 n^{-s[1/p-1/2]}\leqslant\frac{\kappa_{n,s,p}}{\kappa_{n,s,2}}\leqslant K_2 n^{2s[1/p-1/2]},\quad 1\leqslant p\leqslant 2,$$

$$K_3 n^{-2s[1/2-1/p]}\leqslant\frac{\kappa_{n,s,p}}{\kappa_{n,s,2}}\leqslant K_4 n^{s[1/2-1/p]},\quad 2\leqslant p\leqslant\infty,$$

where the constants $K_1,\dots,K_4$ only depend on s.

Proof. By (5.3) we obtain

$$\frac{1}{\binom{n+s}{s}^{1/p-1/2}}\leqslant\frac{\kappa_{n,s,p}}{\kappa_{n,s,2}}\leqslant(K_s n^{2s})^{1/p-1/2},\quad 1\leqslant p\leqslant 2,$$

$$\frac{1}{(K_s n^{2s})^{1/2-1/p}}\leqslant\frac{\kappa_{n,s,p}}{\kappa_{n,s,2}}\leqslant\binom{n+s}{s}^{1/2-1/p},\quad 2\leqslant p\leqslant\infty.$$

For the binomial coefficient we have the upper bound

$$\binom{n+s}{s}=\frac{n^s(1+1/n)(1+2/n)\cdots(1+s/n)}{s!}\leqslant(s+1)n^s$$

and the lower bound $n^s/s!$. The result follows. □

If one looks at the assertion of this corollary one sees that there is still some gap to close. In the univariate case there exists sharper (pointwise) Markov inequalities so one might hope that one could replace $2s$ by s in the factors involving K_2n^{2s} and K_3n^{-2s} in Corollary 9.

The exact behavior of $\kappa_{n,s,p}$ is known in the univariate case. Indeed, in [10] it is shown that

$$\kappa_{n,1,p}2^{-n}n^{1/2p} \to \text{const.}, \quad n \to \infty,\ 1 \leqslant p \leqslant \infty,$$

which means that

$$\frac{\kappa_{n,1,p}}{\kappa_{n,1,2}} = \mathrm{O}(n^{(1/2)(1/p-1/2)}), \quad 1 \leqslant p \leqslant \infty.$$

Thus, we have a positive exponent for $p<2$ and a negative one for $p>2$. But this is not the case in those estimates of Corollary 9 which involve the constants K_1 and K_4.

In the multivariate case we only know the exact behavior for $p=2$. From Theorem 3 we have

$$\kappa_{n,s,2}2^{-n}n^{1/4} \to \text{const}_s., \quad n \to \infty$$

for any fixed space dimension s. However, to guess the exact behavior with respect to the dimension s in the general multivariate case seems difficult. It looks like one needs to find the extremal polynomial in the second sup of (5.1) for $p=1$ and ∞, or have at least have some idea of it.

Appendix. Triangular Legendre polynomials

We start by recalling the definition and some properties of these polynomials. For further properties see [1]. For $\boldsymbol{m}=(m_1,\dots,m_s) \in \mathbb{Z}^s$ with $\boldsymbol{m}\geqslant 0$ we define the Legendre polynomial $P_{\boldsymbol{m}}$ on the standard simplex Σ_s by

$$P_{m_1,\dots,m_s}(x_1,\dots,x_s) = \partial_{x_1}^{m_1}\cdots\partial_{x_s}^{m_s}\left[\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_s^{m_s}}{m_s!}(x_1+\cdots+x_s-1)^{m_1+\cdots+m_s}\right],$$

or more compactly

$$P_{\boldsymbol{m}}(\boldsymbol{x}) = D^{\boldsymbol{m}}\left(\frac{\boldsymbol{x}^{\boldsymbol{m}}}{\boldsymbol{m}!}(|\boldsymbol{x}|-1)^{|\boldsymbol{m}|}\right). \tag{A.1}$$

Clearly $P_{\boldsymbol{m}}$ is a polynomial of degree $|\boldsymbol{m}|$. Indeed, by the multinomial expansion we obtain the explicit representation

$$P_{\boldsymbol{m}}(\boldsymbol{x}) = \sum_{|\boldsymbol{i}|\leqslant|\boldsymbol{m}|}(-1)^{|\boldsymbol{m}|-|\boldsymbol{i}|}\binom{\boldsymbol{m}+\boldsymbol{i}}{\boldsymbol{i}}\begin{bmatrix}|\boldsymbol{m}|\\ \boldsymbol{i}\end{bmatrix}\boldsymbol{x}^{\boldsymbol{i}}, \tag{A.2}$$

where $\left[{|\boldsymbol{m}| \atop \boldsymbol{i}}\right] = |\boldsymbol{m}|!/(\boldsymbol{i}!(|\boldsymbol{m}|-|\boldsymbol{i}|!))$ is a multinomial coefficient and

$$\binom{\boldsymbol{m}+\boldsymbol{i}}{\boldsymbol{i}} = \binom{m_1+i_1}{i_1}\binom{m_2+i_2}{i_2}\cdots\binom{m_s+i_s}{i_s},$$

is a product of binomial coefficients. Consider next orthogonality properties of the Legendre polynomials with respect to the inner product

$$\langle f,g\rangle = \int_{\Sigma_s} f(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Repeated integration by parts shows that for any $f \in C(\Sigma_s)$

$$\langle P_{\boldsymbol{m}}, f\rangle = (-1)^{|\boldsymbol{m}|}\left\langle\frac{\boldsymbol{x}^{\boldsymbol{m}}}{\boldsymbol{m}!}(|\boldsymbol{x}|-1)^{|\boldsymbol{m}|}, D^{\boldsymbol{m}}f\right\rangle \tag{A.3}$$

and it follows that $P_{\boldsymbol{m}}$ is orthogonal to all polynomials of degree $< |\boldsymbol{m}|$

$$\langle P_{\boldsymbol{m}}, f\rangle = 0 \quad \text{for all } f \in P_{\boldsymbol{k}}(\mathbb{R}^s), \quad |\boldsymbol{k}| < |\boldsymbol{m}|. \tag{A.4}$$

In particular, we have $\langle P_{\boldsymbol{m}}, P_{\boldsymbol{k}}\rangle = 0$ for $|\boldsymbol{k}| \neq |\boldsymbol{m}|$, while (A.3) and (A.2) show that $\langle P_{\boldsymbol{m}}, P_{\boldsymbol{k}}\rangle \neq 0$ for $|\boldsymbol{k}| = |\boldsymbol{m}|$. Thus $(P_{\boldsymbol{m}})$ is a sequence of almost orthogonal polynomials.

The degree $|\boldsymbol{m}|$ BB-form of $P_{\boldsymbol{m}}$ is quite simple. Indeed, using Leibniz' rule in each variable separately we have

$$P_{\boldsymbol{m}}(\boldsymbol{x}) = |\boldsymbol{m}|!D^{\boldsymbol{m}}\left(\frac{\boldsymbol{x}^{\boldsymbol{m}}}{\boldsymbol{m}!}\frac{(|\boldsymbol{x}|-1)^{|\boldsymbol{m}|}}{|\boldsymbol{m}|!}\right) = |\boldsymbol{m}|!\sum_{\boldsymbol{k}\leqslant\boldsymbol{m}}\binom{\boldsymbol{m}}{\boldsymbol{k}}\frac{\boldsymbol{x}^{\boldsymbol{k}}}{\boldsymbol{k}!}\frac{(|\boldsymbol{x}|-1)^{|\boldsymbol{m}|-|\boldsymbol{k}|}}{(|\boldsymbol{m}|-|\boldsymbol{k}|)!}.$$

It follows that for any $\boldsymbol{m} \geqslant 0$ the BB-form of the Legendre polynomial is given by

$$P_{\boldsymbol{m}}(\boldsymbol{x}) = \sum_{\boldsymbol{j}\leqslant\boldsymbol{m}}(-1)^{|\boldsymbol{m}|-|\boldsymbol{j}|}\binom{\boldsymbol{m}}{\boldsymbol{j}}B_{\boldsymbol{j}}^{|\boldsymbol{m}|}(\boldsymbol{x}). \tag{A.5}$$

The triangular nature of this relation means that it can be inverted so that we can express the Bernstein basis in terms of the Legendre polynomials, showing that the Legendre polynomials are linearly independent.

The Legendre polynomials are eigenpolynomials for the Gram matrix.

Proposition 10. *For the Gram matrix* $\boldsymbol{G}$ *given by* (3.2) *we have*

$$\boldsymbol{G}\boldsymbol{c}_{\boldsymbol{m}} = \lambda_{\boldsymbol{m}}\boldsymbol{c}_{\boldsymbol{m}}, \quad |\boldsymbol{m}| \leqslant n, \tag{A.6}$$

where $\lambda_{\boldsymbol{m}}$ *is given by* (3.3) *and* $\boldsymbol{c}_{\boldsymbol{m}} = (c_{\boldsymbol{m},\boldsymbol{j}})$ *is the degree n BB-coefficients of the Legendre polynomials* $P_{\boldsymbol{m}}$ *given by* (*A*.1), *i.e.*,

$$P_{\boldsymbol{m}}(\boldsymbol{x}) = \sum_{|\boldsymbol{j}|\leqslant n} c_{\boldsymbol{m},\boldsymbol{j}}B_{\boldsymbol{j}}^n(\boldsymbol{x}). \tag{A.7}$$

Proof. Let $Q_{\boldsymbol{m}}$ be the eigenpolynomials of degree $|\boldsymbol{m}|$ constructed in the proof of Theorem 2. Since $\boldsymbol{G}$ is symmetric, it follows that

$$\langle Q_{\boldsymbol{m}}, Q_{\boldsymbol{k}}\rangle = \boldsymbol{d}_{\boldsymbol{m}}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{d}_{\boldsymbol{k}} = \lambda_{\boldsymbol{k}}\boldsymbol{d}_{\boldsymbol{m}}^{\mathrm{T}}\boldsymbol{d}_{\boldsymbol{k}} = \lambda_{\boldsymbol{m}}\boldsymbol{d}_{\boldsymbol{m}}^{\mathrm{T}}\boldsymbol{d}_{\boldsymbol{k}},$$

and since the eigenvalues $\lambda_{\boldsymbol{m}}$ and $\lambda_{\boldsymbol{k}}$ are distinct for $|\boldsymbol{k}| \neq |\boldsymbol{m}|$ we see that $(Q_{\boldsymbol{m}})$ is a sequence of almost orthogonal polynomials in the sense that

$$\langle Q_{\boldsymbol{m}}, Q_{\boldsymbol{k}}\rangle = 0, \quad \text{for } |\boldsymbol{m}| \neq |\boldsymbol{k}|. \tag{A.8}$$

Since the $(Q_{\boldsymbol{m}})$ are linearly independent we can write each $P_{\boldsymbol{m}}$ in the form

$$P_{\boldsymbol{m}}(\boldsymbol{x}) = \sum_{|\boldsymbol{k}|\leqslant|\boldsymbol{m}|} b_{\boldsymbol{m},\boldsymbol{k}}Q_{\boldsymbol{k}}(\boldsymbol{x}) \tag{A.9}$$

and observe by (A.4) and (A.8) that for each $v < |\boldsymbol{m}|$

$$0 = \langle P_{\boldsymbol{m}}, Q_{\boldsymbol{r}}\rangle = \sum_{|\boldsymbol{k}|=v} b_{\boldsymbol{m},\boldsymbol{k}}\langle Q_{\boldsymbol{k}}, Q_{\boldsymbol{r}}\rangle, \quad \text{for } |\boldsymbol{r}| = v.$$

By taking suitable linear combinations of the $Q_r, |\boldsymbol{r}| = |\boldsymbol{k}|$ we see that this implies that

$$b_{\boldsymbol{m},\boldsymbol{k}} = 0 \quad \text{for } |\boldsymbol{k}| < |\boldsymbol{m}|.$$

It follows that

$$P_{\boldsymbol{m}}(\boldsymbol{x}) = \sum_{|\boldsymbol{k}|=|\boldsymbol{m}|} b_{\boldsymbol{m},\boldsymbol{k}} Q_{\boldsymbol{k}}(\boldsymbol{x})$$

for some numbers $(b_{\boldsymbol{m},\boldsymbol{k}})$. But then the BB-coefficients of $P_{\boldsymbol{m}}$ are a linear combination of the BB-coefficients of $Q_{\boldsymbol{k}}$ for $|\boldsymbol{k}| = |\boldsymbol{m}|$, and these BB-coefficients are eigenvectors of $\boldsymbol{G}$ corresponding to the same eigenvalue. It follows that the degree n BB-coefficient $\boldsymbol{c}_{\boldsymbol{m}}$ of $P_{\boldsymbol{m}}$ is an eigenvector of $\boldsymbol{G}$ corresponding to $\lambda_{\boldsymbol{m}}$.

The linear independence of these eigenvectors follows since both the Legendre polynomials and the Bernstein basis polynomials are bases for the space of polynomials in question.

To summarize: We have shown that for each $\boldsymbol{m}$ with $|\boldsymbol{m}| = n$ the Legendre polynomial $P_{\boldsymbol{m}}$ given by (A.1) is an extremal polynomial for the second sup in (3.1). This case is represented by the smallest eigenvalue of $\boldsymbol{G}$ and the corresponding $\binom{n+s-1}{s-1}$ eigenvectors $\boldsymbol{c}_{\boldsymbol{m}}$ are given by (3.15) which agrees with $|\boldsymbol{m}| = n$ in (A.7) in view of (A.5). The remaining eigenvectors for $|\boldsymbol{m}| < n$ in (A.7) can be determined by degree raising in (A.5). □

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